Global topology of chaotic attractors

Robert Gilmore

Physics Department, Drexel University, Philadelphia, PA, USA 19104
robert.gilmore@drexel.edu

Résumé

A great deal is known about the global topological structure of chaotic attractors in three dimensional spaces, but little is known about the spectrum of topologies allowed to chaotic attractors in higher dimensions. We review some of the tools that have contributed to this understanding in three dimensions and show how these tools can be applied in higher dimensions. Specifically, we show how singularities of mappings can be used to classify chaotic attractors in generalized tori $D^N \times S^1$ where stretching and folding are the mechanisms creating chaos. We show how symmetry arguments have been used to create chaotic attractors that are generated by tearing and squeezing mechanisms. We introduce families of locally diffeomorphic chaotic attractors that are described by quantum numbers in R^3 and show that these quantum indices are group labels in higher dimensions. Simple classical measures are introduced to help describe chaotic attractors in any dimension.

1 Introduction

The global topological structure of strange attractors $\mathcal{SA}$ in three dimensions is more or less known in some detail [1,2,3] for two broad reasons. At the intuitive level, we can visualize such attractors rather easily, sometimes even by closing our eyes and using our imaginations. At the mathematical level there is a theorem — the Birman-Williams theorem [4] — that allows us to represent every dissipative chaotic attractor by a surrogate — a two-dimensional branched manifold. These surrogates can all be built up Lego© style by plugging together two types of units — stretch and squeeze units — in any discernible way provided two elementary conditions are met. In higher dimensions intuition fails and theorems fail to exist.

In order to understand the global topology of chaotic attractors in higher dimensions we must rely on analogies with the structure of attractors in three-dimensional spaces. For this reason we begin with a review of the properties of these low dimensional attractors. Broadly speaking there are two types : (1) those created by the infinite repetition of stretching and folding ; and (2) those created by infinite repetition of stretching, tearing, and squeezing. The first class of attractors live inside a torus of genus $g = 1 : D^2 \times S^1$. The latter live inside higher genus tori : $g \geq 3$ [5,6]. We exploit this decomposition to get a grip on the spectrum of chaotic attractors that we can expect to encounter in higher-dimensional spaces.

2 Three Dimensions

The Birman-Williams theorem is the most powerful tool at our disposal for the study of chaotic attractors in three dimensions. A consequence of this theorem is that a chaotic flow (in fact, its surrogate, a branched manifold) in R^3 can be decomposed into a union of two types of units : stretch units and squeeze units, shown in Fig. 2(A). These units can be plugged together in any way provided only that (1) outflows are connected to inflows and (2) there are no free ends. Each such union is a branched manifold that represents a chaotic attractor. Four such branched manifolds are shown in Fig. 2(B). We notice that three of these $((a) - (c))$ live inside a torus of genus $g = 1$ and of these, one (c) lives in a toral annulus. The fourth (d) lives inside a higher-genus $(g = 3)$ torus.

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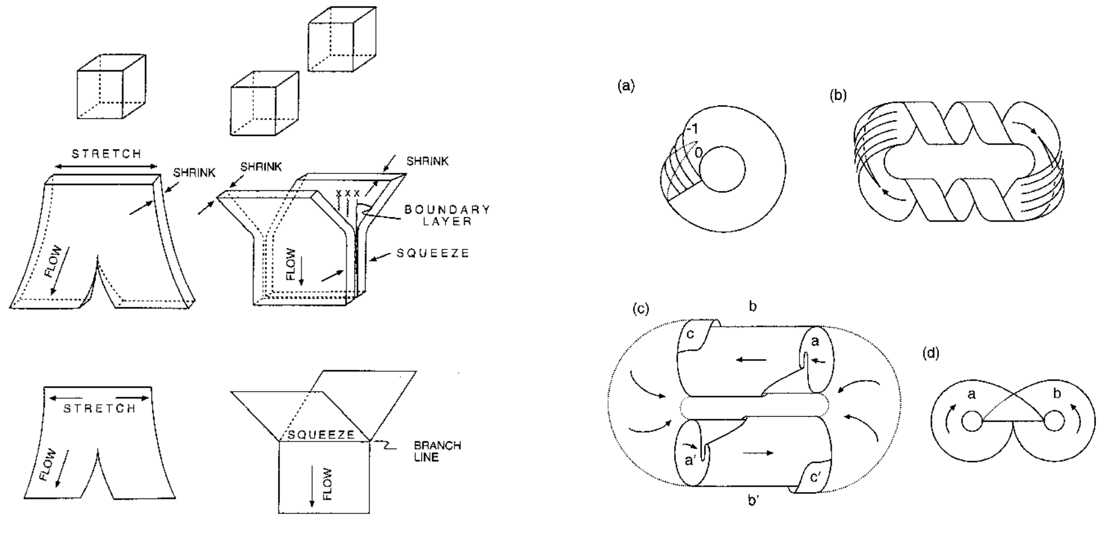


(A) (B)

Fig.1. (A) Stretching and squeezing units that are used to build up an arbitrary branched manifold. Cf. [1], Fig. 5.3 (B) Branched manifolds for four standard strange attractors. (a) Rössler attractor ; (b) Duffing attractor ; (c) van der Pol attractor ; (d) Lorenz attractor. Cf. [1], Fig. 4.1

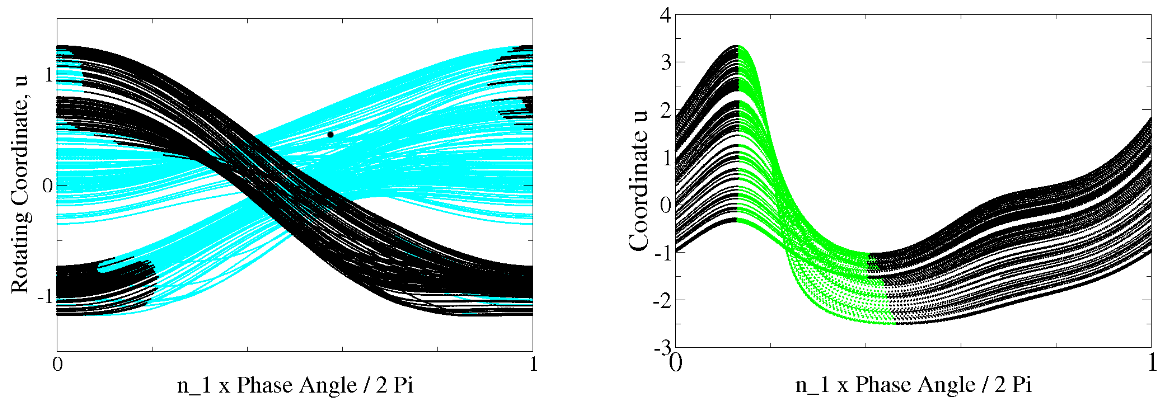
3 $3D$, $g = 1$, $D^2 \times S^1$

We concentrate in this Section on three dimensional chaotic attractors that live in a torus of genus one (solid tire). Most of the experimental data sets that can successfully be embedded in three dimensions share this property [7].

When this is the case it is useful to construct a Poincaré surface of section, $\mathcal{PS}$. This is a surface whose intersection with the torus is a disk, D^2. The flow is usefully treated as a return map of the Poincaré section to itself : more specifically of $\mathcal{PS} \cap \mathcal{SA}$ to itself. By combining this with the results of the Birman-Williams theorem, we see that the entropy-generating mechanism is the return map of $\mathcal{PS} \cap \mathcal{BM}$, where $\mathcal{BM}$ is the branched manifold that describes the strange attractor. This intersection is a one-dimensional set that can be chosen without singularities : either an interval I or a circle S^1. The only possible singularities of maps of these one-dimensional sets to themselves are singularities of fold type : A_2. If the return map $I \rightarrow I$ has n fold singularities the corresponding branched manifold has $n+1$ branches, where $n \geq 1$. Global topology has a nontrivial impact on local singularities : for S^1, n must be even.

It is possible to represent the coordinates of a point in a genus-one strange attractor in the form (X, Y, ϕ),with $(X, Y) \in D^2$ and $\phi \in S^1$. For the Duffing and van der Pol nonautonomous systems $\phi = \omega t$ mod 2π. For autonomous sytems such as the Rössler system a similar representation is possible for standard control parameter values, using $x + iy = Ae^{i\phi}$ and $(X, Y, \phi) = (A, dA/d\phi, \phi)$. Plots of the Duffing and Rössler attractor in this toroidal representation are shown in Fig. 3.

In this representation the attractor itself has 2π periodicity. We can exploit the periodicity of attractors in tori of genus one to construct entire families of strange attractors closely related to the original attractor [8]. This is done as follows. Create a diffeomorphism of the strange attractor by rotating the coordinates (X, Y) in the plane ϕ through an angle $\theta = \theta(\phi)$. Periodicity of the strange attractor requires $\theta(\phi = 2\pi) = \theta(\phi = 0) + 2\pi n_2$, where n_2 is an integer. Such a diffeomorphism twists the strange attractor through $2\pi n_2$ radians over the length of the torus. The original attractor and its image are not isotopic unless $n_2 = 0$. Since the mapping is a diffeomorphism and since all fractal dimensions and Lyapunov dimensions are diffeomorphism invariants, all members in this family ($n_2 = \cdots, -2, -1, 0, +1, +2, \cdots$) have identical



(A) Duffing attractor, subharmonic lift
Index $(n_1, n_2) = (1, 0)$

(B) Rössler attractor, Toroïdal representation
Index $(n_1, n_2) = (1, 0)$

Fig.2. Toroidal representation of (A) the Duffing attractor and (B) the Rössler attractor.

spectra of fractal dimensions and Lyapunov exponents. Members of the Duffing family with $n_2 = -1, +1$ and of the Rössler family with $n_2 = -1, +1$ are shown in Fig. 3.

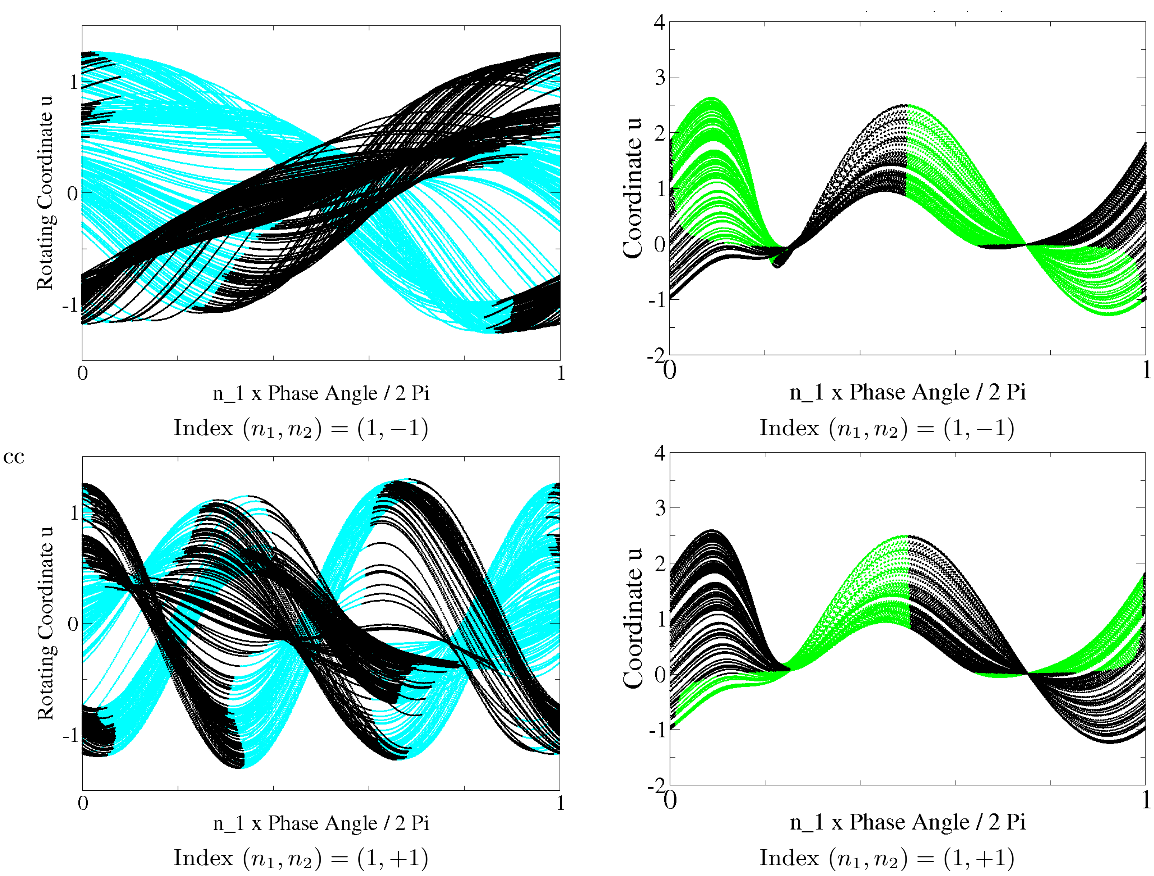
There are two real-valued measures that are suitable for distinguishing among members in these families of strange attractors. These measures are average values of the energy- and angular momentum-like integrals. For example, the energy measure is $\langle E \rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{1}{2}(\dot{X}^2 + \dot{Y}^2)dt$, with the angular momentum average defined analogously. For diffeomorphisms generated by uniform rotations $\theta(\phi) = n_2\phi$ these integrals behave exactly as one would expect from experience with elementary classical mechanics. These integrals are shown for the family based on the Duffing attractor in Fig. 4. The energy and angular momentum integrals for subharmonic lifts of the Duffing attractor are shown in Fig. 3.

If the "quantization condition" $\theta(\phi = 2\pi) = \theta(\phi = 0) + 2\pi n_2$ is not satisifed (i.e., $n_2 \neq$ integer) then harmonic lifts are not possible. If the condition $\theta(\phi = 2\pi) = \theta(\phi = 0) + 2\pi n_2/n_1$ is satisfied, periodic boundary conditions are satisfied for $\omega t = 2\pi n_1$ and the attractor closes up with a period $n_1 T_1$. When it closes up the plane $(\mathcal{PS})$ containing the intersection of the attractor has rotated through $2\pi n_2$ radians. These are *subharmonic lifts* of the founding member of the family of attractors. Subharmonic lifts are indexed by *two* relativeluy prime quantum numbers (n_1, n_2), $n_1 \geq 1$. Two subharmonic lifts of the Rössler attractor are shown in Fig. 3.

4 $3D$, $g > 1$

Some well-known strange attractors are created by repetition of the stretching-tearing-squeezing mechanism. The prototypical example is the Lorenz attractor. Three dimensional attractors generated by this mechanism live in bounding tori of genus g with $g > 1$. In R^3 closed bounded two-surfaces are described by a single integer, the genus g (Intrinsic description). However, these surfaces are dressed by the flow that generates the strange attractor within. As a result more than a single index g is required to distinguish bounding tori that contain inequivalent strange attractors with the same genus [2,3]. Fig 7 shows the five inequivalent canonically dressed tori of genus 7 that can contain strange attractors [9]. Also shown in this figure are three ways to label these canonical tori as well as the transition matrices that describe how the flow can progress from one component of the Poincaré surface of section to other components. The Poincaré surface of section is the union of $g - 1$ disks, each bounded by a meridian of the torus. Every three-dimensional strange attractor is bounded by one of these tori, as shown in Table 1.

Chaotic attractors in some of the lower-g bounding tori can be constructed by lifting chaotic attractors in a genus-one bounding tori using some simple group theoretical arguments [9]. This construction fails



(A) Duffing attractor, subharmonic lift (B) Rössler attractor, toroïdal representation

Fig.3. Toroidal representation with $n_2 = -1$ (top) and $n_2 = +1$ (bottom) of members of (A) the Duffing family and (B) the Rössler family of attractors. Darker : $Y > 0$; Lighter : $Y \leq 0$.

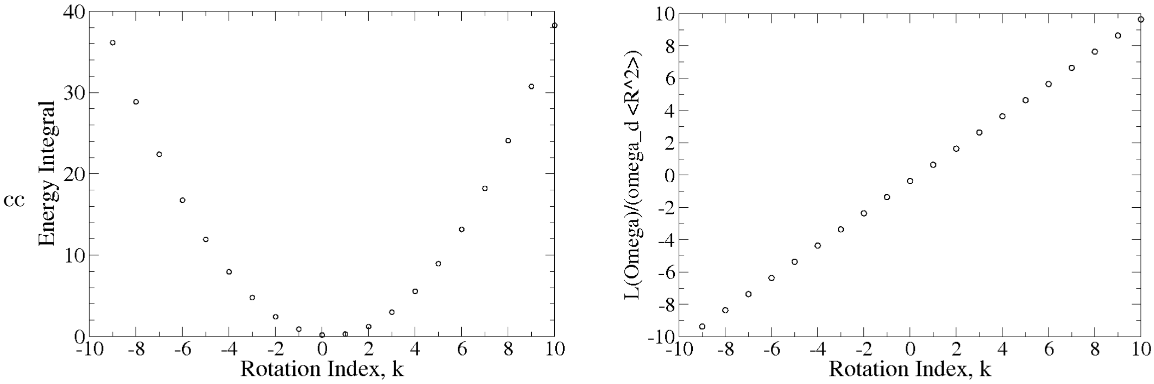
when the bounding torus does not have sufficient symmetry. In this case other lifting mechanisms must be used. The result is a set of $(g-1) \rightarrow 1$ local diffeomorphisms between the lifted attractor and the attractor in a genus-one bounding torus. In a very real sense we create "covering attractors" in genus-g bounding tori by creating "topological lifts" from a simpler attractor, one generated by simple stretching-and-folding processes.

Since two-parameter families of chaotic attractors can be created from any strange attractor, there is a very large variety of chaotic attractors even in the low-genus cases. It is not yet known how to classify them all.

The number of canonical bounding tori grows rapidly with the genus g, as shown in Table 2. This number, $N(g)$, increases exponentially with g : $N(g) \simeq e^{h_g g}$. In a computational tour-de-force Katriel [10] has shown that $h_g = \log(3.0)$. This author believes that $h_g = \log(3)$ for the same reason that the *topological* entropy of the logistic map is $\log(2)$, not $\log(2.0)$.

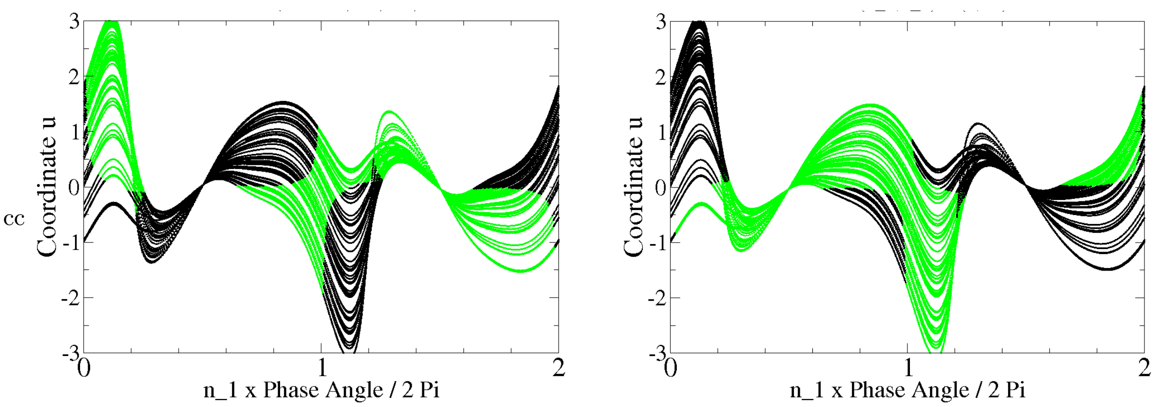
5 Higher Dimensions

In N dimensions there are N Lyapunov exponents $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$. These split into three groups : the n_u positive Lyapunov exponents that describe unstable motion (stretching), the zero exponent(s), which describes the flow direction, and the negative exponents that describe the squeezing directions. We will assume that only one Lyapunov exponent is zero : λ_{n_u+1}. Associated with each Lyapunov exponent



(A) Energy integral (B) Torsion integral

Fig.4. Classical (A) energy and (B) torsion integrals in the Duffing family of harmonic attractors. Plots show dependence on the quantum number n_2.

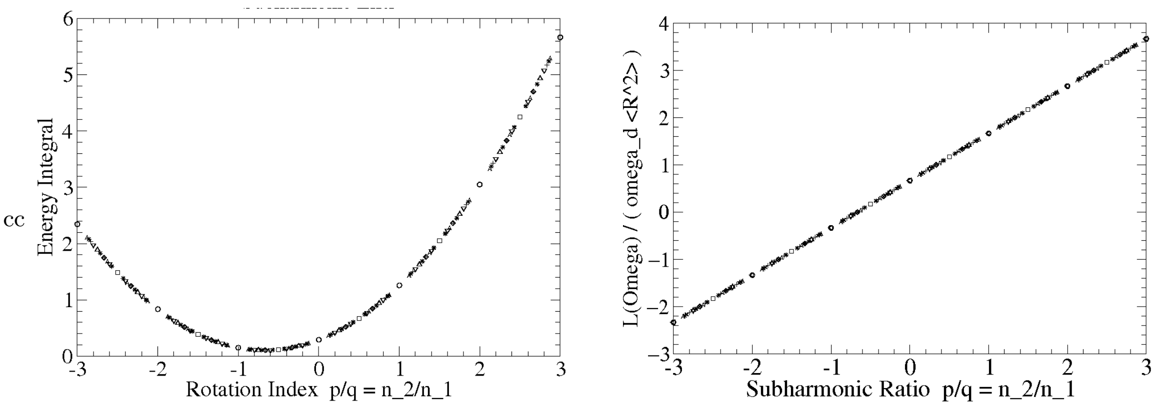


(A) Index $(n_1, n_2) = (2, -1)$ (B) Index $(n_1, n_2) = (2, +1)$

Fig.5. Subharmonic lifts of the Rössler attractor with quantum numbers $(n_1, n_2) =$ (A) $(2, -1)$ and (B) $(2, +1)$.

there is a partial dimension $d_1 \geq d_2 \geq \cdots \geq d_N$, with $1 \geq d_i \geq 0$ [11] . The partial dimensions are +1 for the expanding and flow directions with $\lambda_i \geq 0$: the strange attractor is smooth in these directions. In the squeezing directions, in which the attractor has a fractal structure, the partial dimensions are generally less than one.

In dimensions greater than 3 much less is known about the topology of strange attractors. There is the hope that a pair of theorems can be found to simplify our understanding of such attractors. Both depend on the Lyapunov exponents of the attractor. The first theorem would identify the dimension, K, of an "embedding manifold" $\mathcal{EM}$ containing the strange attractor. Specifically, The weighted sum $D_j = \sum_{i=1}^{j} \lambda_i d_i$ is formed. This sum generally increases with j, levels off at $j = n_u + 1$ when $\lambda_{n_u+1} = 0$, and decreases to zero thereafter. Define K to be the smallest integer for which $D_K = D_{K+1} = \cdots D_N = 0$. It would be nice to have a theorem stating that a K-dimensional manifold can be found that contains the strange attractor. At an intuitive level, this means that the flow is very strongly attracted to the embedding manifold along the directions with the most strongly contracting eigendirections (those with the most negative Lyapunov exponents : $\lambda_{K+1}, \lambda_{K+2}, \cdots, \lambda_N$). Once inside the embedding manifold the flow never leaves it. Further, information about the most negative Lyapunov exponents is no longer available : information about only the $n_s = K - (n_u + 1)$ "weakly negative" exponents $\lambda_j, j = n_u + 2, \cdots, K$ is available. Here n_u is the number of unstable directions (positive Lyapunov exponents) and n_s is the number of stable directions within the embedding manifold.



(A) Energy integral (B) Torsion integral

Fig.6. Classical (A) energy and (B) torsion integrals in the Duffing family of subharmonic attractors. Plots show dependence on the quantum numbers (n_1, n_2) is through their ratio n_2/n_1.

Tab.1. All known strange attractors of dimension $d_L < 3$ are bounded by one of the standard dressed tori.

Strange Attractor	Dressed Torus	Period $g-1$ Orbit
Rossler, Duffing, Burke and Shaw	A_1	1
Various Lasers, Gateau Roule	A_1	1
Neuron with Subthreshold Oscillations	A_1	1
Shaw-van der Pol	$A_1 \cup A_1^{(1)}$	$1 \cup 1$
Lorenz, Shimizu-Morioka, Rikitake	A_2	$(12)^2$
Multispiral attractors	A_n	$(12^{n-1})^2$
$\mathcal{C}_n$ Covers of Rossler	C_n	1^n
$\mathcal{C}_2$ Cover of Lorenz$^{(a)}$	C_4	1^4
$\mathcal{C}_2$ Cover of Lorenz$^{(b)}$	A_3	$(122)^2$
$\mathcal{C}_n$ Cover of Lorenz$^{(a)}$	C_{2n}	1^{2n}
$\mathcal{C}_n$ Cover of Lorenz$^{(b)}$	P_{n+1}	$(1n)^n$
$2 \rightarrow 1$ Image of Fig. 8 Branched Manifold	A_3	$(122)^2$
Fig. 8 Branched Manifold	P_5	$(14)^4$

$^{(a)}$ Rotation axis through origin.
$^{(b)}$ Rotation axis through one focus.

Tab.2. Number of canonical bounding tori as a function of genus, g.

g	$N(g)$	g	$N(g)$	g	$N(g)$	g	$N(g)$	g	$N(g)$
1	1	5	2	9	15	13	368	17	14290
2	0	6	2	10	28	14	870	18	36824
3	1	7	5	11	67	15	2211	19	96347
4	1	8	6	12	145	16	5549	20	252927

The second theorem would guarantee the existence of an $n_u + 1$-dimensional "manifold with singularities" resulting from a Birman-Williams-like projection. Such a projection identifies all points with the same future : $x \simeq y$ if $|x(t) - y(t)| \stackrel{t\rightarrow\infty}{\longrightarrow} 0$. This projection mods out the n_s-dimensional stable manifold (the "weakly negative" Lyapunov eigendirections) over any point in the strange attractor. Such a theorem would allow us to identify strange attractors with their singular limits under the projection : a process that has made the classification and analysis of strange attractors in R^3 possible.

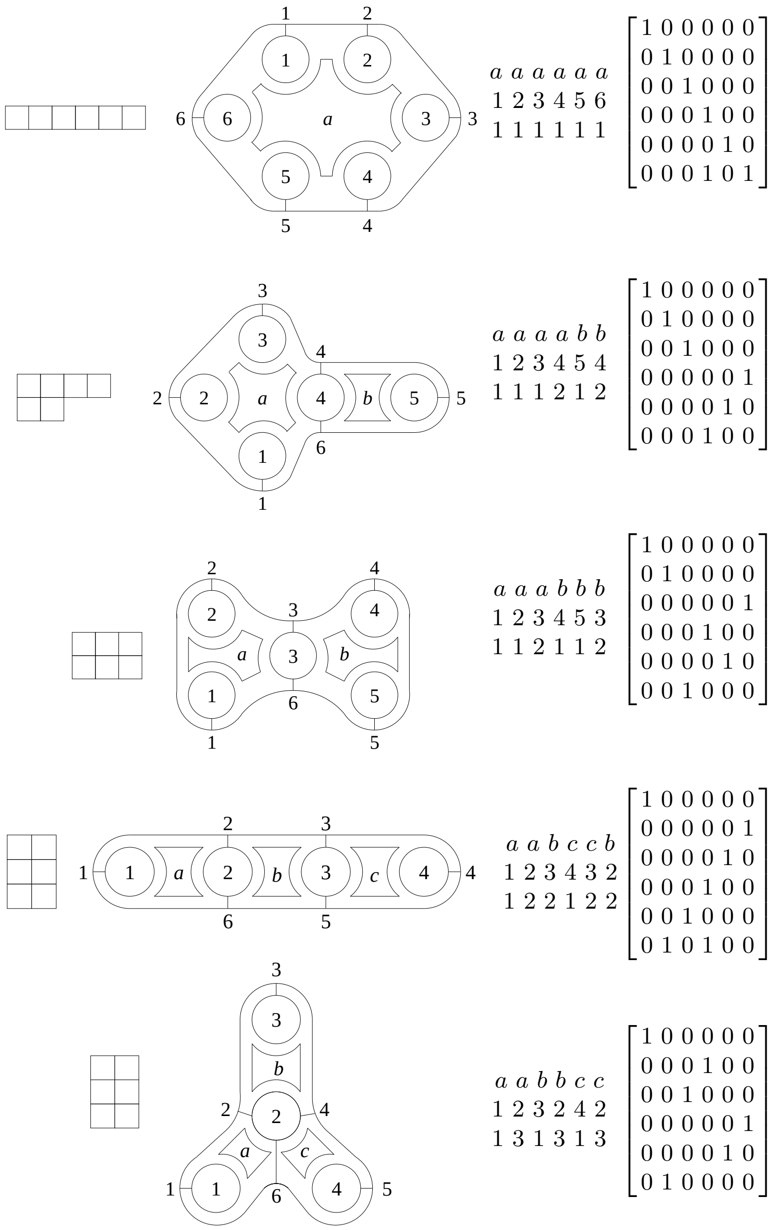


Fig.7. There are five canonically dressed tori of genus 7. Two inequivalent flows have Young partitions 2^3.

6 $D > 3$, $g = 1$, $D^{n_u+n_s} \times S^1$

We assume first that the embedding manifold is a higher-dimensional torus $D^{K-1} \times S^1 = D^{n_u+n_s} \times S^1$. In this case the Poincaré surface of section is a constant phase slice of the torus : $\phi = \text{cnst.}$, $0 \leq \phi < 2\pi$. The slice is an $n_u + n_s$-dimensional space $D^{n_u+n_s}$. The flow is characterized, modulo some group theoretical indices, by a return map. If the second theorem, or some useful version of it, is true, the return map can be treated as a mapping of a n_u-dimensional space to itself. In Fig. 8 we show mappings of one dimensional manifolds ($n_u = 1$) and two dimensional manifolds ($n_u = 2$) to themselves. These mappings must generate entropy : therefore they must have singularities. The two mappings shown generate the fold (A_2) and the cusp (A_3) singularities, respectively.

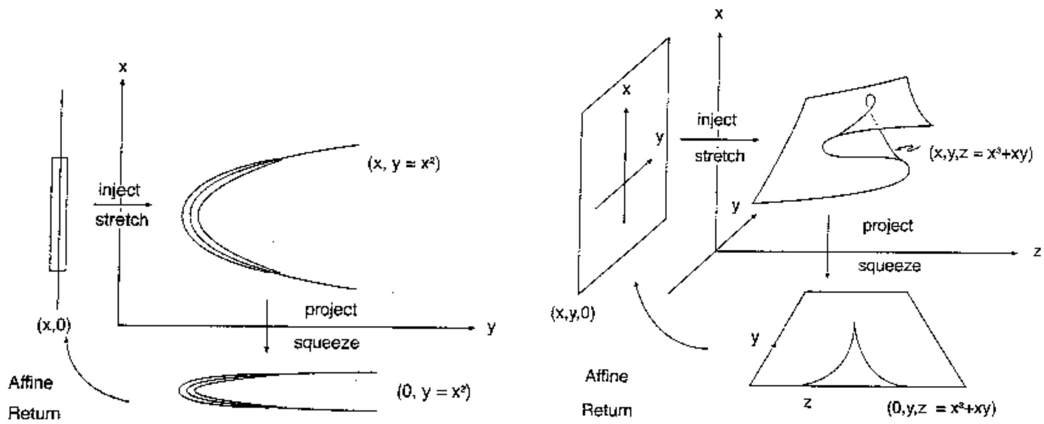


Fig.8. (left) Intersection of a strange attractor in $R^2 \times S^1$ with a Poincaré section is almost an interval. The Poincaré return map exhibits a fold singularity, producing a logistic map. (right) Intersection of a strange attractor with two positive Lyapunov exponents in $R^3 \times S^1$ with a Poincaré section is almost a plane section. The Poincaré return map exhibits a cusp singularity. Cf. [1], Fig. 11.4, Fig. 11.5.

Under the assumptions outlined above (two provable theorems), we can make inroads on the problem of describing higher dimensional strange attractors by describing the kinds of singularities that can occur in mappings of k dimensional manifolds to themselves in spaces of dimension $K-1$. In Fig. 8 the *stable* singularities of mappings of the interval to itself in R^2 (left) and of the plane to itself in R^3 (right) are shown. In R^3 the double fold singularity $(x, y) \rightarrow (x^2, y^2)$ is also possible but it is not stable, it perturbs to a cusp and a fold singularity.

The study of stable singularities of mappings $R^k \rightarrow R^k$ has a long history [16]. They are described by Young partitions Λ with row lengths $(\lambda_1, \lambda_2, \cdots, \lambda_r)$ and $\lambda_{i+1} \leq \lambda_i$. The Young partitions for the fold and cusp maps in Fig. 8 are (1) and $(1,1)$. For the cusp $\lambda_1 = 1$ indicates that the mapping has rank one less than full (2) on the fold lines and $\lambda_2 = 1$ tells us that, on the fold set the mapping drops by one again at the cusp point (the origin). In general the row length y_1 describes how much the rank of the mapping drops on the singularity of largest dimension; y_2 describes by how much the mapping drops in rank at the second singularity when restricted to the first singularity, etc. The singularity may be stable under perturbations $A_2 : x \rightarrow x^2$ or may be unstable under perturbations $A_n : x \rightarrow x^n,\ n > 2$. In the latter case for $n = 3$ the singularity $(x, y) \rightarrow (x^3, y)$ perturbs to $(x, y) \rightarrow (x^3 + xy, y)$, which is structurally stable.

For the purposes at hand the singularities we wish to consider have three parts : a germ, a universal perturbation (also called an unfolding), and folding directions. As an example, the two-dimensional germ $(x, y) \rightarrow (x^3 + y^2, xy)$ has a three-dimensional unfolding with basis vectors $(x, y, x^2) : D_G = 2, D_U = 3$ [16]. The structurally stable mapping $R^{2+3} \rightarrow R^5$ is $(x_1, x_2; u_1, u_2, u_3) \rightarrow (x_1^3 + x_2^2 + u_1 x + u_2 y + u_3 x_1^2, xy, u_1, u_2, u_3)$. In order for the folding to occur smoothly in the space $D^{n_u+n_s}$ it is necessary that there be at least two (= dimension of the germ) additional "folding" directions $(f_1, f_2) : D_F = 2$. In short, the singularity $(x, y) \rightarrow (x^3 + y^2, xy)$ is first encountered in flows in $D^{n_u+n_s} \times S^1$ for $n_u \geq D_G + D_U = 5$, $n_s \geq D_F = D_G = 2$, $n_u + n_s \geq 7$. When this inequality is saturated $(n_u + n_s = 7)$ a discrete classification may be possible.

In summary, flows in K dimensional embedding manifolds can be classified by germs of dimension D_G with structurally stable unfoldings of dimension D_U provided $D_G + D_U \leq n_u$ and $n_s \geq D_F \geq D_G$. For the cuspoids A_n with $x \rightarrow x^n$, $D_G = 1, D_U = n-2$, $D_F = 1$ and $n_u \geq 1 + (n-2)$ $(n \geq 2)$ and $n_s \geq D_F = 1$. In three dimensions the only stable singularity is the fold : $n_u = 1, n_s = 1$, and one time flow direction.

The construction introduced in Section 3 can be applied to strange attractors in the torus $D^{n_u+n_s} \times S^1$ in an attempt to create families of globally or locally diffeomorphic strange attractors. The idea is the same : the results are different. Strange attractors in the torus $D^{n_u+n_s} \times S^1$ are periodic, with period 2π. That is, the attractor is invariant under $\phi \to \phi + 2\pi$, where ϕ is the angular coordinate in S^1. New attractors $\mathcal{SA}'$ are created from an attractor $\mathcal{SA}$ by a ϕ-depentent rotation $\theta(\phi)$. Specifically, we apply a rotation $R(\hat{\mathbf{n}}, \theta)$ to each section in such a way that periodic boundary conditions are satisfied : $R(\hat{\mathbf{n}}, \theta(\phi = 0)) = R(\hat{\mathbf{n}}, \theta(\phi = 2\pi))$. For simplicity we keep the rotation axis $\hat{\mathbf{n}}$ fixed during this process. This creates a new attractor from the original. The two are globally diffeomorphic. As a result their spectra of fractal dimensions and Lyapunov exponents are identical, and cannot be used to distinguish one from another. On the other hand, classical measures, such as average energy and angular momentum can be used to help distinguish among the members of a family. As the axis $\hat{\mathbf{n}}$ around which the rotation takes place is varied over the sphere surface, the energy integral also varies. These energy bands for a strange attractor in $R^3 \times S^1$ are shown in Fig. 9.

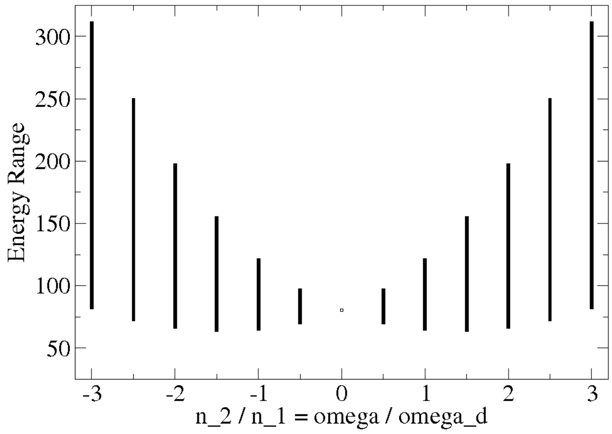


Fig.9. The Rössler equations were periodically driven to create a strange attractor in $R^3 \times S^1$. The basic member of this family was used to create an entire family of strange attractors by satisfying periodic boundary conditions. The energy depends on the orientation of the rotation axis $\hat{\mathbf{n}}$, held fixed during the rotation. The range of energy values is shown here.

The next question to address is whether the harmonic and subharmonic lifts of a founding member of a family of strange attractors $\mathcal{SA} \subset D^{n_u+n_s} \times S^1$ are all topologically inequivalent, as is the case for familites in $D^2 \times S^1$.

Transformations of period T depending on the phase $\phi(t)$ in the interval $0 \leq t \leq T$ can be interpreted as paths in the parameter space of the Lie group $SO(N)$ $(N = n_u + n_s)$. Two paths in the parameter space of $SO(N)$ starting at $\phi_j(0) = 0$ $(j = 1, 2)$ and ending at integer values (to satisfy periodic boundary conditions) $\phi_j(T)/T = n_j$ can or cannot be deformed into each other depending on whether $n_1 - n_2$ is even or odd. When $N > 2$, the fundamental group of $SO(N)$ has two elements, those corresponding to rotations through $2\pi n$ radians with n even (n_e) or with n odd (n_o). Once again, the path $\phi(t)$ can be chosen as linear, but this time inequivalent paths exist with $n = 0$ or $n = 1$ only. For the deformation in $SO(3)$ that takes a path with $n = 2$ to the path with $n = 0$ see [14].

As a result, any strange attractor in $D^N \times S^1$ $(N > 2)$ can form the base for only four diffeomorphic but topologically inequivalent chaotic attractors. These are labeled by indices from two two-element groups : a parity index $\rho = \{e, o\}$ from $O(N)/SO(N)$, and an index $\sigma = \{n_e, n_o\} = \{0, 1\}$ labeling an element in the fundamental group of $SO(N)$:

$$\mathcal{SA} \stackrel{\rho e^{n\omega L t}}{\longrightarrow} \mathcal{SA}_{\rho,\sigma} \quad (1)$$

This classification can be extended to subharmonic transformations. The procedure follows the steps indicated in Sect. 3, with some subtle differences. For the fundamental group operation n_e there are no subharmonic attractors. Subharmonics exist only for the fundamental group operation n_o. The result leads to two-parameter families of chaotic attractors that are locally diffeomorphic but not globally diffeomorphic ($p > 1$) with the original attractor. The members of these families are indexed by two discrete indices : the parity index ρ and a positive integer p related to the period pT at which the proper boundary conditions are satisfied :

$$\mathcal{SA} \overset{\rho \exp((1\omega/p)(\hat{\mathbf{n}}\cdot\mathbf{L})t)}{\longrightarrow} \mathcal{SA}_{\pm 1, n_o, p} \tag{2}$$

As a result of this analysis, we conclude that far fewer chaotic attractors can be constructed by rotation transformations in higher dimensions than in three dimensions.

7 $D > 3$, $g > 1$

The strange attractors in tori $D^{K-1} \times S^1$ are created by smooth processes : stretching and folding, whether $K - 1 = 2$ or $K - 1 > 2$. In three dimensions we understand how to relate strange attractors in higher-genus tori from strange attractors in the simple genus-one torus. The construction involves : many-one maps, singularities, group theory, symmetry and, when symmetry is insufficient, topology.

It is possible to use the same procedures in higher dimensions to construct attractors in R^K that do not live in a toroidal manifold $D^{K-1} \times S^1$. This has been done both for nonautonomous and autonomous four dimensional flows.

The nonautonomous flow is a periodically driven Rössler attractor, whose phase space is $R^3 \times S^1$. This was mapped into R^4 using the natural embedding : $(x_1, x_2, x_3, \omega t) \to (y_1, y_2, y_3, y_4)$, with $y_1 = x_1, y_2 = x_2$ and $y_3 = (a - x_3)\cos\omega t, y_4 = (a - x_3)\sin\omega t$. The radius a was chosen so that $a - x_3(t) > 0$ for all t. The autonomous four-dimensional flow generated a strange attractor in a torus $D^3 \times S^1$. The treatment of both attractors from this point on was the same.

A Lorenz-like attractor can be created by constructing a double cover of the Rössler attractor in the usual way [12,13,9]. If (x_1, x_2, x_3) are coordinates in a Rössler attractor, a double cover with coordinate (z_1, z_2, z_3) is created using the $2 \to 1$ mapping $x_1 = z_1^2 - z_2^2, x_2 = 2x_1x_2, x_3 = z_3$. In the present case four-fold covers with coordinates (z_1, z_2, z_3, z_4) were created using the $4 \to 1$ maps (paired double covers) : $y_1 = z_1^2 - z_2^2, y_2 = 2z_1z_2$ and $y_3 = z_3^2 - z_4^2, y_4 = 2z_3z_4$. These four-fold lifts from $D^3 \times S^1 \to R^4$ are shown in Fig. 10.

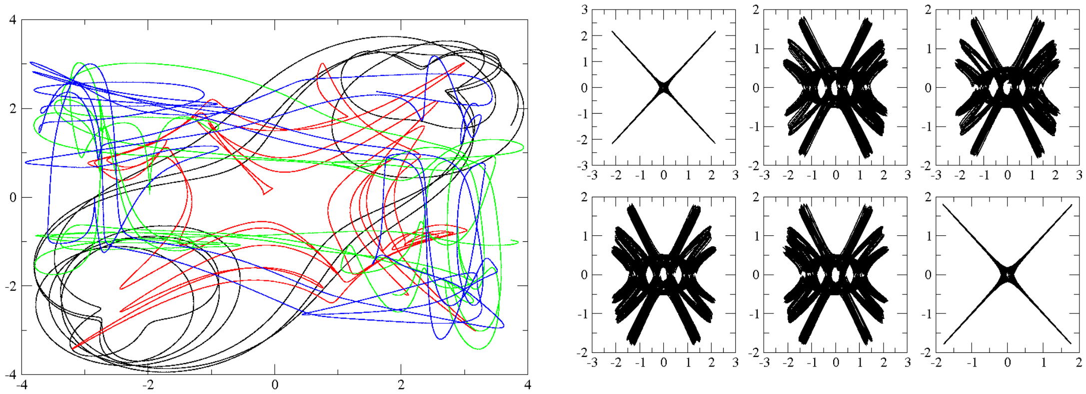


Fig.10. Four-fold covers of periodically driven Rössler equations (left) and a four dimensional autonomous dynamical system generating a strange attractor in $D^3 \times S^1$.

8 Conclusions

We have constructed families of topologically inequivalent strange attractors in three dimensional tori $D^2 \times S^1$ based on a single strange attractor (periodically driven Duffing, van der Pol, Takens-Bogdanov attractors, Rössler attractor). Family members are indexed by three quantum numbers (ρ, n, p). Members of each family are either isotopic to, globally diffeomorphic with, or locally diffeomorphic with the original attractor, as indicated :

Diffeo. Type	Parity	Fund. Gp.	Subharmonic
Isotopic to Id	$\rho = +1$	$\sigma = 0$	$p = 1$
Global	$\rho = \pm 1$	$\sigma = n$	$p = 1$
Local	$\rho = \pm 1$	$\sigma = n \neq 0$	$p > 1$

(3)

with n and p relatively prime in the last line. Each quantum number has a group-theoretical interpretation. This construction has been extended to attractors in $N+1$ dimensional solid tori $D^N \times S^1$, with result summarized below :

Diffeo. Type	Parity	Fund. Gp.	Subharmonic
Isotopic to Id	$\rho = +1$	$\sigma = n_e$	$p = 1$
Global	$\rho = \pm 1$	$\sigma = n_e, n_o$	$p = 1$
Local	$\rho = \pm 1$	$\sigma = n_o$	$p > 1$

(4)

In this case three indices are still required to identify members of each family. They are all group operations : $\rho \in O(N)/SO(N) \simeq Z_2$, σ is a member of the fundamental group of $SO(N)$, also equivalent to Z_2, and $p \geq 1$ is a homotopy index of $SO(2)$. Since the standard real measures (Lyapunov exponents and fractal dimensions) are invariant under global and local diffeomorphisms, they are useless for distinguishing different members of a family. We have introduced two classical mechanics statistics for distinguishing among them. These are an average energy integral and an average angular momentum integral about the rotation axis. Both statistics depend systematically on the angular frequency ω' and are simple to compute from time series.

The biggest losses in going from strange attractors in $D^2 \times S^1$ to strange attractors in $D^N \times S^1$ ($N > 2$) are loss of the global torsion index n (from $SO(2)$) [15] and its replacement by a Z_2 valued index σ (from $SO(N)$), and the greatly reduced number of topologically inequivalent families that can be constructed from diffeomorphisms that satisfy periodic boundary conditions after 1 or p periods.

Acknowledgements : The author thanks M. Lefranc and C. Letellier for helping to organize this Colloquium, for exciting interactions, and useful comments.

Références

1. R. Gilmore & M. Lefranc, *The Topology of Chaos*, Wiley, New York (2002).
2. T. D. Tsankov & R. Gilmore, Strange attractors are classified by bounding tori, *Phys. Rev. Lett.*, **91**(13), 134104 (2003).
3. T. D. Tsankov & R. Gilmore, Topological aspects of the structure of chaotic attractors in R^3, *Physical Review E*, **69**, 056206 (2004).
4. J. Birman & R. F. Williams, Knotted periodic orbits in dynamical systems II : Knot holders for fibered knots, *Cont. Math.*, **20**, 1-60 (1983).
5. G. Byrne, R. Gilmore & C. Letellier, Distinguishing between folding and tearing mechanisms in strange attractors, *Physical Review E*, **70**, 056214 (2004).
6. C. Letellier, T. D. Tsankov, G. Byrne & R. Gilmore, Large scale structural reorganization of strange attractors, *Physical Review E*, **72**, 026212 (2005).
7. R. Gilmore, Topological analysis of chaotic dynamical systems, *Revs. Mod. Phys.*, **70** (4), 1455-1529 (1998).
8. R. Gilmore, Two-parameter families of strange attractors *Chaos*, **17**, 1 (2007).
9. R. Gilmore & C. Letellier, *The Symmetry of Chaos*, Oxford University Press, New York, 2007.
10. J. Katriel & R. Gilmore, Entropy of bounding tori, (unpublished).
11. J.-P. Eckmann and D. Ruelle, Ergodic theory of chaos and strange attractors, *Revs. Mod. Phys.*, **57** (3), 617-656 (1985).
12. R. Gilmore & C. Letellier, Covering dynamical systems : Twofold covers, *Physical Review E*, **63**, 016206 (2001).
13. C. Letellier & R. Gilmore, Dressed symbolic dynamics, *Physical Review E*, **67**, 036205 (2003).
14. R. Gilmore, *Lie Groups, Lie Algebras, and Some of their Applications*, Wiley : NY (1974).
15. H. G. Solari & R. Gilmore, Relative rotation rates in driven dynamical systems, *Phys. Rev. A*, **37**, 3096-3109 (1988).
16. V. I. Arnol'd, A. Varchenko & A. Gousein-Zadé, *Singularitiés des applications differentiales*, Editions Mir, Moscou, 1986.