The spectrum of large powers of the Laplacian in bounded domains

E. Katzav[1] & M. Adda-Bedia[1]

Laboratoire de Physique Statistique de l'Ecole Normale Supérieure, CNRS UMR 8550
24 rue Lhomond, 75231 Paris Cedex 05, France
eytan.katzav@lps.ens.fr

Recently there has been a growing interest in the problem of finding the spectrum of large powers of the Laplacian in bounded domains [1,2,3]. In one dimension, which is the case that will interest us here, the problem is simply that of finding eigenfunctions and eigenvalues to the equation

$$(-\Delta)^N u\,(x) = \lambda u\,(x) \qquad\qquad x \in [-1,1]\ , \tag{1}$$

for functions obeying the following Boundary Conditions (BC)

$$u\,(\pm 1) = u^{(1)}\,(\pm 1) = \cdots = u^{(N-1)}\,(\pm 1) = 0\,, \tag{2}$$

where $u^{(k)}(x)$ is the k^{th} derivative of $u(x)$.

From a mathematical point of view [1] the determination of the spectrum of Δ^N can be related to four different problems - the spectrum of certain positive definite Toeplitz Matrices, the norm of the Green kernels of Δ^N, the best constants in certain Wirtinger-Sobolev inequalities, and the conditioning of a special least squares problem. From a physical point of view, the interest in the spectrum of Δ^N comes from many directions. Classical problems such as diffusion and wave propagation require knowledge of the spectrum of Δ (i.e., $N = 1$), which is also related to recent problems such as diffusion limited aggregation and chaos. Problems in elasticity theory often deal with Δ^2, and the fractional Laplacian (i.e., when N is not necessarily an integer) appears naturally in stochastic interfaces and Lévy flights [4,5] and so knowledge of the spectrum allows progress in the understanding of anomalous diffusion and first-passage problems of a Lévy flyer.

We begin [6] with the one-dimensional case and show that the whole spectrum can be obtained in the limit of large N. More precisely, the eigenfunction of Δ^N with absorbing BC can be written as certain associated Legendre polynomials, namely

$$v_j(x) \propto P_{2N+j}^{2N}\,(x) + O\left(1/N^2\right) \qquad \text{for} \quad j = 0, 1, 2, \ldots\,. \tag{3}$$

up to proper normalization, and the eigenvalues are just

$$\lambda_j = (-1)^N \sqrt{2}\,(2N)!\frac{(4N)^{2j}}{(2j)!}\left[1 - \frac{3+4j+8j^2}{16N} + O\left(\frac{1}{N^2}\right)\right]\,. \tag{4}$$

Actually, Our approach allows us to obtain systematical corrections to the eigenvalues and eigenvectors as a power series where $1/N$ plays the role of the small parameter, and we do so up to order $1/N^4$.

It turns out that this basis is a good choice for diagonalizing the differential operator Δ^N also when N is not large, which implies a useful numerical approach valid for any N. Finally, we discuss implications of this work and present its possible extensions for non-integer N and for $3D$ Laplacian problems.

Références

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