Auto-résonance de l’instabilité Raman stimulée due à une non-linéarité d’origine cinétique dans un plasma inhomogène

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Abstract. New solutions to the coupled three-wave equations describing parametric scattering instabilities are presented. This analytical and numerical study investigates the impact of autoresonance on stimulated Raman scattering in an inhomogenous plasma, where the dominant plasma wave nonlinearity is due to wave-particle interactions (a kinetic-type nonlinearity). Under conditions in the plasma relevant to laser fusion, it is shown that the electron plasma wave may become phase-locked to the beating between a laser pump wave and a back-scattered light wave. This phase-locked solution may grow to a significant amplitude. It is shown that the growth of the electron plasma wave will lead to a break-down of the phase-locking and hence its maximum attainable amplitude is inherently limited.

1 Introduction

Stimulated Raman Scattering (SRS) in warm plasmas is a parametrically unstable resonant three-wave interaction, where laser light (the pump wave) scatters off an electron plasma wave (the Langmuir wave), resulting in a second electromagnetic wave (the scattered wave). The understanding of the SRS process is of central importance to the realisation of laser thermonuclear fusion. Through back-scattering of the laser and fast-electron generation, SRS reduces the efficiency of the laser light absorption that heats the plasma corona. This in turn inhibits the necessary ablation and compression processes and may eventually inhibit the ignition of the thermonuclear deuterium-tritium fuel.

In the presence of a nonlinear perturbation such as wave-profile steepening or kinetic effects, the Langmuir wave can be considered as a nonlinear oscillator coupled to the electromagnetic waves [1]. The frequency of a nonlinear oscillator varies with the amplitude of oscillation. Starting from rest, if the system is driven at its linear (constant) frequency, then the amplitude of oscillation will initially increase. However, the oscillator will quickly dephase from the drive and the amplitude will subsequently beat back to a small value. If instead the system is driven at a frequency that also varies, the oscillator may, under certain conditions, phase-lock to the drive and automatically stay resonant, allowing the amplitude to grow well beyond the response to a fixed-frequency drive.

Autoresonance arises in many systems such as driven pendulums [2], electron beams in accelerators [3,4] and superconducting Josephson junctions [5]. Yaakobi \emph{et al.} [1] investigated the phenomenon of autoresonance in the SRS process in a physical regime where the dominant nonlinear frequency shift (NFS) of the oscillation (in this case, in the Langmuir wave equation) arose from thermal and relativistic effects in the plasma. This fluid-type NFS was proportional to the squared Langmuir wave amplitude.
In the following work, we investigate autoresonance in the SRS process in the physical regime where the dominant NFS arises from kinetic effects. This kinetic-type NFS is proportional to the square root of the Langmuir wave amplitude [8].

2 The three-wave equations with a kinetic-type nonlinearity

We consider the case where the incident electromagnetic pump wave (forwards-propagating) couples to the Langmuir wave (forwards-propagating) and a back-scattered electromagnetic wave (backwards-propagating) in a plasma with a density that increases linearly in the forwards direction. The SRS process can be described by the following system of quadratically-coupled envelope equations (see e.g. Ref. [6]), relevant to a one-dimensional, weakly non-uniform, stationary, underdense, thermal plasma:

\[
\begin{align*}
\left( \frac{\partial}{\partial t} + c_0 \frac{\partial}{\partial z} \right) a_0 &= -\frac{\omega_p^2}{2\epsilon_0} a_1 a_L, \\
\left( \frac{\partial}{\partial t} - |c_1| \frac{\partial}{\partial z} \right) a_1 &= \frac{\omega_p^2}{2\omega_1} a_0 a_L^*, \\
\left( \frac{\partial}{\partial t} + c_L \frac{\partial}{\partial z} + i\beta|a_L|^{1/2} - ic_L\kappa'z \right) a_L &= \gamma\omega_0 a_0 a_1^*.
\end{align*}
\]

We describe each of the three waves in terms of a slowly-varying complex envelope and a quickly-varying phase. The envelopes \(a_0\) and \(a_1\) describe the pump and scattered waves respectively and the Langmuir wave is described by \(a_L = \delta n_e/n_0\) where \(n_0\) is the plasma density before the formation of the Langmuir wave and \(\delta n_e\) is the perturbation of this density.

We assume the three-wave resonance condition \(\omega_0 = \omega_1 + \omega_L\) for the frequencies, while the wave vectors \(k_{0,1,L}\) are slowly varying functions of space \(z\). The SRS process is relevant in the regime where all frequencies are greater than \(\omega_p/2\), where the plasma frequency \(\omega_p^2 = \omega_p^2(z) = n_0 e^2/\epsilon_0 m_e\), for which \(\epsilon_0\) is the permittivity of free space, \(e\) is the charge of the electron and \(m_e\) is the electron mass. The wave vectors are mismatched according to \(k_0 - k_1 - k_L = \kappa'z\), where \(\kappa'\) parameterises the spatial inhomogeneity (derived by Liu et al. [7]), and act to detune the three waves from resonance either side of \(z = 0\). The vacuum speed of light is given by \(c\) while the group velocities of the three waves are \(c_{0,1,L}\). The coupling strength of Langmuir wave to the pump and scattered waves is given by the parameter \(\gamma = (k_L^2 c^2/2\omega_L\omega_0)(I_F/n_e mc^2)\) where \(I_F\) is the incident pump wave energy flux and \(n_e\) is the density up to which the pump wave can penetrate (the critical density).

The frequency shift of the Langmuir wave \(-ic_L\kappa'z\) in equation (3) is due to the mismatch in the wave vectors, with the spatial resonance occurring at \(z = 0\). This study focuses on the kinetic NFS \(i\beta|a_L|^{1/2}\) in equation (3) and its potential to balance the frequency shift due to the wave vector mismatch over a significant distance beyond the resonance point. The strength \(\beta\) of the kinetic NFS is assumed constant in all simulations.

3 Analysis using a prescribed ponderomotive drive

We begin this investigation of spatial autoresonance by initially reducing the system of three coupled equations to a single equation describing the Langmuir wave. We prescribe the amplitudes of the electromagnetic waves on the RHS of equation (3) and replace \(\gamma\omega_0 a_0 a_1^*\) with a prescribed ponderomotive drive \(P\). This allows the investigation of spatial autoresonance to be uncomplicated by effects due to the depletion of the pump wave or growth of the scattered wave. Furthermore, the Langmuir wave is driven by a single frequency meaning SRS may occur only at the resonance point.

For convenience of analysis, we define the characteristics \(t = s - s_*\) and \(z = c_L s\), where \(s\) is a real variable chosen so that \(s = 0\) at \(z = 0\) and \(s_*\) is a constant. The characteristics are straight lines in the \((z,t)\)-plane, intercepting the \(z\)-axis at \(s_*\) and bound by \(z_L\) and \(z_R\) (the left and right boundaries of...
the plasma respectively). Using the total derivative of $a_L$, we may rewrite equation (3) as the ordinary differential equation

$$\frac{da_L}{ds} + i(\beta|a_L|^{1/2} - c_L^2\kappa's)a_L = P(s),$$

which is readily solved along these characteristics. Equation (4) must then be solved for a range of values of $s$, with the initial condition $a_L(z = z_L) = 0$ in order for the space-time dependence of the Langmuir wave equation to be recovered. For $t < 0$, $P(s) = 0$ and $a_L$ is undriven. At $t = 0$, we switch on the prescribed ponderomotive drive. For $t \geq 0$, $P(s)$ is assigned a fixed value, uniform throughout the plasma hence $P(s) = \mu H(s - s_*)$, where $\mu$ is a constant and $H$ is the Heaviside step function.

Following the results of Yaakobi et al. [1], we look for solutions to equation (4) characterised by the autoresonant growth of $a_L$ to high amplitude, where the phase $\phi_L = \arg(a_L)$ is constant over a region in space (i.e. where $a_L$ is phase locked to $P$). We also expect solutions characterised by $\phi_L$ changing rapidly and $a_L$ oscillating around a constant value. In Fig. 1, we present a solution to equation (4), showing the propagation and evolution of $a_L$ through space and time. For consistency, the values of $\beta$ and $P$ were chosen to be in agreement with the parameters used in section 4 ($P$ is in this case real and negative due to the locked phases $\phi_0 = \arg(a_0) \approx 0$ and $\phi_1 = \arg(a_1) \approx -\pi$, calculated in section 4). The solution is autoresonant with a growing wave front closely following the curve

$$|a_L| = (c_L^2|\beta'|z)^2 \quad \text{or} \quad \beta|a_L|^{1/2} = c_L^2\kappa's$$

for $z, s > 0$, indicating a cancellation of the kinetic NFS and the frequency shift due to the plasma inhomogeneity. The phase $\phi_L \approx \pi/2$ throughout the region behind the wave front. Behind the wave front, $a_L$ is steady in space and time but for small fluctuations.

The solutions presented by Yaakobi et al. [1] for the growth of the Langmuir wave in a regime where the dominant NFS is of a fluid type displayed a pronounced threshold phenomenon. In this case, the Langmuir wave was observed to dephase quickly from the drive when driven below threshold. When driven above this threshold, however, the phase-locking of the Langmuir wave to the drive continued indefinitely, allowing the Langmuir wave to grow arbitrarily high if pump depletion effects were not introduced. This behaviour was explained by Fajans and Friedland [2] by drawing analogy to a driven pendulum and considering the formation of pseudo-potential wells that act to trap the phase of the pendulum, leading to small oscillations of $\phi_L$ around a constant value. A similar analysis of theses pseudo-potential wells where a kinetic NFS replaces the fluid NFS may be performed. By substituting $a_L = |a_L|\exp(i\phi_L)$ into equation

Fig. 1. (Colour online) Solutions to the prescribed ponderomotive drive Langmuir wave equation (4) showing the growth and propagation of the Langmuir wave through a positive plasma density gradient at different times, $t = 0.9, 1.7, 2.5, 3.3$ ps. The equation $|\delta n_0/n_0| = \alpha z^2$ where $\alpha = (c_L^2\kappa'/|\beta|) = 8 \times 10^{-5}$ describes the growth of the leading edge of the Langmuir wave, based on the cancellation of the kinetic nonlinear frequency shift and the frequency shift due to the wave vector mismatch.
we may write
\[ \frac{dI}{ds} = -2I^{1/2}|P| \cos(\phi_L), \]
\[ \frac{d\phi_L}{ds} = c_L^2 \kappa' s - \beta I^{1/4} + I^{-1/2}|P| \sin(\phi_L), \]
where \( I = a_L^2 \) is the action of the system. From solutions to equation (4), we observe the phase to be locked at \( \phi_L \approx \pi/2 \) as the system passes through resonance at \( s = 0 \). For the system to remain autoresonant and for the drive \( P \) to be effectively coupled to \( I \), the phase must remain locked at this value. We separate \( I \) into the slowly-varying average action \( I_0 \) and a perturbation \( \Delta \), where \( I = I_0 - \Delta \). Since \( \phi_L \) is constant during autoresonance, we may set its derivative in equation (7) to zero and using the average action write
\[ 0 \approx c_L^2 \kappa' s - \beta I_0^{1/4} + I_0^{-1/2}|P|. \]
Differentiating this expression and solving for \( I_0 \), we find
\[ \frac{dI_0}{ds} = \frac{c_L^2 \kappa'}{M}, \]
where the slowly-varying parameter \( M = M(s) \) is
\[ M = \frac{|P|}{2I_0^{3/2}} + \frac{\beta}{4I_0^{3/4}}. \]
We now expand equations (6) and (7) around the instantaneous value of the average action \( I_0 \) by substituting equation (9) into equation (6) and equation (8) into equation (7). To lowest order, we find
\[ \frac{d\Delta}{ds} = 2I_0^{1/2}|P| \cos(\phi_L) + \frac{c_L^2 \kappa'}{M}, \]
\[ \frac{d\phi_L}{ds} = \Delta M. \]
Equations (11) and (12) form a Hamiltonian system \( \mathcal{H} = \mathcal{H}(\Delta, \phi_L, s) \) where
\[ \mathcal{H} = \frac{M \Delta^2}{2} - \left( 2I_0^{1/2}|P| \sin(\phi_L) + \frac{c_L^2 \kappa'}{M} \phi_L \right) = \frac{1}{2M} \left( \frac{d\phi_L}{ds} \right)^2 + V(\phi_L). \]
We may interpret equation (13) as a Hamiltonian governing the behaviour of a pseudoparticle at \( s \approx 0 \) that describes the phase \( \phi_L \) of the Langmuir wave. This pseudoparticle has a slowly-varying effective mass \( M \) and travels through a potential \( V \) that varies with \( \phi_L \). The potential \( V = V_{\text{lin}} + V_{\text{osc}} \) is a sum of two terms: a linear term \( V_{\text{lin}} = -(c_L^2 \kappa'/M) \phi_L \) and a series of potential wells \( V_{\text{osc}} = -2I_0^{1/2}|P| \sin(\phi_L) \). Under the condition that \( M \) varies slowly, these wells may be significant and can act to trap the pseudoparticle provided that
\[ \frac{|dV_{\text{osc}}|}{d\phi_L} > \frac{|dV_{\text{lin}}|}{d\phi_L}, \]
for a range of values of \( \phi_L \) within the period of \( V_{\text{osc}} \). In this case, the slope of \( V_{\text{lin}} \) is small enough that the oscillation of \( V_{\text{osc}} \) causes an overall oscillation in the sign of the gradient of \( V \). This will be true while
\[ 2I_0^{1/2}|P| > \frac{c_L^2 \kappa'}{M}. \]
Since we have assumed that the system is initially autoresonant, we begin with the pseudoparticle trapped in \( V \) and \( \phi_L \) performing small oscillations around \( \pi/2 \). Initially, the average action \( I_0 \) grows according to \( I_0 \approx |a_L|^2 \propto s^4 \). However, as \( I_0 \) increases, we see from equation (10) that \( M \) decreases quickly and that the RHS of equation (15) increases faster than the LHS of equation (15). This results in a weakening of the pseudoparticle trapping and the eventual loss of autoresonance, shown in Fig. 2.
Fig. 2. (Colour online) The potential $V = -(e^2 \kappa' \beta / M) \phi_L - 2 |I_0|^{1/2} |P| \sin(\phi_L)$ shown at $I_0 = 1 \times 10^{-5}$ (solid red line) and $I_0 = 8.1 \times 10^{-4}$ (dashed green line) corresponding to $z \approx 0 \mu m$ and $z \approx 35 \mu m$ in Fig. 1 respectively. All parameters used to calculate $V$ other than $I_0$ are identical in the two cases. The potential wells are capable of trapping the phase of the Langmuir wave $\phi_L$ while $2 |I_0|^{1/2} |P| > e^2 \kappa' \beta / M$. The potential wells become less significant and eventually disappear as the average action $I_0$ increases, ending the autoresonant region in space. For clarity, $V$ has been increased by a factor of 100 for $I_0 = 1 \times 10^{-5}$ (solid red line).

The sharp threshold behaviour of autoresonance arising from a fluid NFS found by Yaakobi et al. [1] is not observed in the kinetic case. Regardless of the parameters chosen, autoresonance where the dominant NFS is kinetic will always eventually be lost as a (or $z$), and consequently $I_0$, increases. In Fig. 1, we observe this breakdown of phase-locking and the resulting plateau of the amplitude at $|\delta n_e/n_0| = 0.09$. From equation (15), it is clear that this maximum amplitude increases with the magnitude of both $\beta$ and $P$. However, due to the $\beta^{-2}$ dependence of the Langmuir wave amplitude in equation (5), the wave front will grow more slowly in space as $\beta$ is increased and require a longer distance to reach a plateau. Equation (15) may be solved for $I_0$, and using unchanged parameters, we find a predicted maximum amplitude $|\delta n_e/n_0| = 0.06$. This is lower than the actual maximum reached in the solutions shown in Fig. 1 since the growth of $\phi_L$ will still be slow for a short distance in space after the initial loss of the potential wells, resulting in a continued increase in $|\delta n_e/n_0|$.

4 Three-wave coupling code with a positive density gradient

Using a finite difference method, we numerically solve the space-time dependent system of equations (1-3) between the boundaries $z_L$ and $z_R$. At $z_R$, we seed the system by injecting a back-scattered wave from a noise source (here, we use a Langevin equation) with a broad frequency spectrum, allowing SRS to occur at all points in the plasma. At $z_L$, a pump wave is injected after ensuring that the injected back-scattered wave has propagated throughout the plasma. To better approximate the physical situation, we include a nonlinear Landau damping term $\nu = 0.03 \omega_p [1 + \beta |a_{L,max}|^{1/2} (t/\tau)]^{-1}$ in equation (3), where $\tau$ is the time for resonant trapped electrons to bounce across the potential well of the wave and $a_{L,max}$ is the maximum local value of the Langmuir wave which propagates at the phase velocity $c_{ph} \equiv \omega_L/k_L$.

In Fig. 3 we present solutions to equations (1-3), assuming a peak pump wave intensity $I_P = 5 \times 10^{15}$ W/cm$^2$ with wavelength $\lambda_0 = 2 \pi c/\omega_0 = 351$ nm, and an injected back-scattered wave at an average thermal level corresponding to an intensity of approximately $5 \times 10^4$ W/cm$^2$. The inhomogeneity parameter $\kappa' = 4.4 \times 10^7$ cm$^{-2}$. The wave vector of the Langmuir wave $k_L$ and the Debye length $\lambda_D$ are such that the plasma parameter $k_L \lambda_D (z = 0) \approx 0.34$ is in a typically “kinetic” regime.

In Fig. 3(a), $a_L = \delta n_e/n_0$ is shown as a function of $z$ at a range of times showing the evolution of the Langmuir wave. We find that the leading edge of the Langmuir wave front follows the relation found using a prescribed ponderomotive drive given in equation (5). In Fig. 3(b), the phase is observed to be locked at $\phi_L \approx \pi/2$ over a distance that increases with the same speed that the wave front of $a_L$ propagates. The significant deviation of the phase from this value occurs simultaneously to the collapse of the steady form of the Langmuir wave front. In Fig. 3(c), the effective cancellation between the kinetic NFS and the
5 Conclusion

Under conditions relevant to laser fusion experiments, the autoresonant SRS-driven Langmuir wave obtained by solving the three-wave equations has a slowly-changing form with a wave front that grows according to $|\delta n_e/n_0| = \alpha z^2$, where $\alpha = (c_L \kappa'/\beta)^2 = 8 \times 10^{-5}$ μm$^{-2}$ describes the growth of the leading edge of the Langmuir wave, based on the \cancellation of the kinetic nonlinear frequency shift and the frequency shift due to the wave vector mismatch. (b) the phase $\phi_L$ associated with the Langmuir wave. (c) the magnitude of the kinetic NFS term in the wave equation of the Langmuir wave (normalised to the term describing the frequency shift due to the wave vector mismatch)

Fig. 3. (Colour online) Solutions to the three-wave equations (1-3). (a) the growth and propagation of the Langmuir wave is shown at different times, $t=2.9, 3.5, 4.1, 4.3$ ps. The equation $|\delta n_e/n_0| = \alpha z^2$ where $\alpha = (c_L \kappa'/\beta)^2 = 8 \times 10^{-5}$ μm$^{-2}$ describes the growth of the leading edge of the Langmuir wave, based on the cancellation of the kinetic nonlinear frequency shift and the frequency shift due to the wave vector mismatch. (b) the phase $\phi_L$ associated with the Langmuir wave. (c) the magnitude of the kinetic NFS term in the wave equation of the Langmuir wave (normalised to the term describing the frequency shift due to the wave vector mismatch)

shift due to the spatial inhomogeneity may be seen. The cancellation is total only for a short distance in space at the leading edge of the Langmuir wave growth.

Références