Dispersion relation for unidirectional surface gravity waves

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1-D focusing Nonlinear Schrödinger (NLS) equation is often considered as a leading order model that governs dynamics of unidirectional surface gravity waves. Studies of the wind-generated ocean waves revealed that the surface elevation has Gaussian distribution which is a result of the superposition of a large number of harmonics with random phases [1,2]. Such random wave (or partially-coherent wave) can be considered as an initial condition for a nonlinear problem governed by 1-D NLS. Together random initial conditions and integrable equation bring us to the field known as Integrable Turbulence [3].

In the present work, we introduce a novel point of view on Integrable Turbulence describing the random wave at the stationary state by corresponding dispersion relation. We write 1-D focusing NLS in the following way:

$$i\frac{\partial A}{\partial z} = \frac{1}{g}\frac{\partial^2 A}{\partial t^2} + k_0^3|A|^2 A, \tag{1}$$

where g is the acceleration of gravity, k_0 is a wave number. Complex envelope A is related to the surface elevation η as $\eta(z,t) = \frac{1}{2}\left(A(z,t)e^{i(k_0 z-\omega_0 t)} + c.c.\right)$, where $\omega_0 = 2\pi\nu$, with ν the central frequency. Therefore the dispersion relation in the case of partially-coherent waves can be written as:

$$\tilde{k}(\omega,\epsilon) = \omega^2/g - 2\omega_0^2\epsilon^2/g \tag{2}$$

The first term on the right-hand side is the linear dispersion relation, while the second one is nonlinear correction where $\epsilon = k_0\sqrt{<|A|^2>}$ is the steepness.

We provide a numerical investigation of the dynamics of the partially-coherent waves solving the Eq. 1 with a standard pseudo-spectral method. In the weakly nonlinear regime, when dynamics is dominated by dispersive waves, the dispersion relation is simply represented by a parabola with a coefficient $1/g$. Increasing nonlinearity, we found that the shift in k is, indeed, linearly proportional to ϵ^2 until a certain value of nonlinearity ($BFI = \epsilon/(\Delta\omega/\omega_0) > 0.34$). For the higher values of BFI we observe the appearance of straight lines in $k(w)$ curve. The lines have a certain distribution in angles and positions. We show that each line corresponds to a fundamental soliton. The slope is related to the group velocity and displacement in k to its amplitude. For a large number of BFI, solitonic content dominates and dispersive waves are no longer represented in the $k(\omega)$ curve. However, we discovered that the growth of the position of the center of mass remains linear with respect to the ϵ^2, but with another coefficient of proportionality.

We verify these results by providing water tank experiments. The investigated zone of parameters varies from steepness 0.06 to 0.14 with the carrying wave frequency 1.15 Hz and the spectral width is 0.2 Hz. Surface elevation measurements were provided in 20 equidistant points separated by 6 m. Experimental results are in a good agreement with simulations of focusing 1-D NLS equation up to the steepness 0.1. In the case of more nonlinear waves, we observe significant asymmetry which signifies that the higher order terms have to be included in the model.

References

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