Wave turbulence in self-gravitating Bose gases and nonlocal nonlinear optics

Jonathan Skipp
j.m.skipp@warwick.ac.uk

Victor L'vov

Sergey Nazarenko

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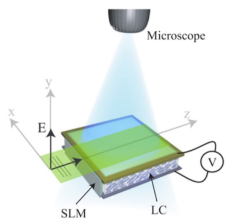
Schrödinger-Helmholtz equation

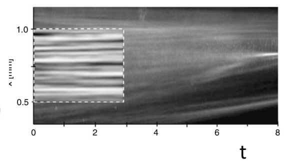
Aim – Understand dynamical formation of large-scale structures:

3D: Galactic haloes in ultralight dark matter ($m\sim10^{-22}$eV)

NASA, ESA, S. Beckwith (STScI) & Hubble Heritage Team

2D: Condensates in nonlocal nonlinear optics





Laurie et al, Phys Rep (2012)

Both systems described by the **Schrödinger-Helmholtz equation** (SHE)

$$i\partial_t\psi + \nabla^2\psi - \psi V = 0 \qquad (\nabla^2 - \Lambda)V = \gamma|\psi|^2$$

- ψ: Boson wavefunction / Beam envelope
- V: Gravit'l potential / Refractive index
- ∇^2: Newtonian gravity / Optical nonlocality
- Λ: Cosmological constant / Kerr coefficient

This poster reports 3D case with $\Lambda = 0$, see paper for other limits and for 2D.

Wave turbulence of the SHE

Wave turbulence: ensembles of random, weakly-nonlinear waves.

Wave specrtum $n_\mathbf{k} \sim \langle|\hat{\psi}_\mathbf{k}|^2\rangle$ evolves according to **kinetic equation:** $(\omega_\mathbf{k} = k^2)$

$$\partial_t n_\mathbf{k} = 4\pi \int \left|W^{12}_{3\mathbf{k}}\right|^2 \delta^{12}_{3\mathbf{k}}\delta\left(\omega^{12}_{3\mathbf{k}}\right) n_1 n_2 n_3 n_\mathbf{k} \left[\frac{1}{n_1} + \frac{1}{n_2} - \frac{1}{n_3} - \frac{1}{n_\mathbf{k}}\right] d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3$$

Adiabatic invariants particles $N = \int n_\mathbf{k}\, d\mathbf{k}$, energy $E = \int \omega_\mathbf{k} n_\mathbf{k}\, d\mathbf{k}$

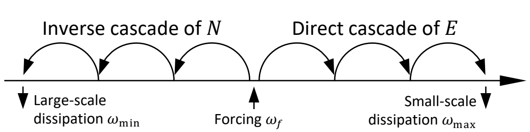
Rayleigh-Jeans (RJ) spectrum thermal equilibrium $\quad n^{RJ}_\mathbf{k} = \dfrac{T}{\mu + \omega_\mathbf{k}}$

Particle equipartition $n^{TN}_\mathbf{k} \propto k^0$
Energy equipartition $n^{TE}_\mathbf{k} \propto k^{-2}$

Kolmogorov-Zakharov (KZ) spectra flux of invariants $\quad n^{KZ}_\mathbf{k} \sim k^{-x}$

Particle cascade $n^{FN}_\mathbf{k} \propto k^{-1}$
Energy cascade $n^{FE}_\mathbf{k} \propto k^{-5/3}$

Fjørtoft argument:
Dual cascade of invariants



Cascade directions on KZ spectra

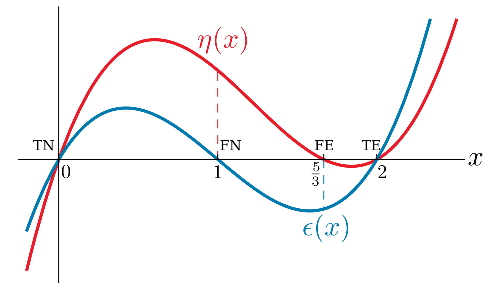
We plot the flux of particles η and energy ϵ for power-law spectra $n_\mathbf{k} \sim k^{-x}$

Steep spectra: fluxes adjust to flatten steep spectra.

RJ spectra: $\epsilon = \eta = 0$ on $n^{TN}_\mathbf{k}, n^{TE}_\mathbf{k}$ (no flux in equilibrium).

KZ spectra: $\epsilon = 0$ on $n^{FN}_\mathbf{k}$, $\eta = 0$ on $n^{FE}_\mathbf{k}$ (spectra of pure N and E flux respectively).

Continuity of η, ϵ in between.



The fluxes on both KZ spectra are in the wrong direction c.f. Fjørtoft argument.

Differential approximation model

To resolve the contradiction we simplify the wave kinetic equation by assuming $\omega_\mathbf{k} \approx \omega_1 \approx \omega_2 \approx \omega_3$, deriving a **differential approximation model**

$$\partial_t\left(\omega^{1/2} n\right) = \partial_{\omega\omega} R \quad , \quad R = \omega^{9/2} n^4 \partial_{\omega\omega}(1/n).$$

Assuming a weakly-perturbed thermal spectrum $n = \frac{T}{\mu+\omega+\theta(\omega)}$ yeilds

$$\eta = -\frac{15\omega^{3/2}_{\min}}{4}\left(\frac{T}{\mu}\right)^3 \quad , \quad \epsilon = \frac{3T^3}{4\omega^{1/2}_{\max}}$$

Conclusion: large-scale structure is constructed by a **warm inverse cascade** i.e. a nearly-thermal spectrum that carries particles from the forcing to the condensate scale.

Similar result for 2D case with relevance to optics.