Paths to synchronization

A. España[1,2], X. Leoncini[1], & E. Ugalde[2]

[1]Centre de Physique Théorique, Aix-Marseille Université, Campus de Luminy, 163 Avenue de Luminy, 13009 Marseille, France.
[2]Instituto de Física, Universidad Autónoma de San Luis Potosí, Av. Manuel Nava 6 Zona Universitaria, 78290, San Luis Potosí, Mexico.

andreae@mail.ifisica.uaslp.mx

Synchronizing systems

This work shows a new approach to the study of dynamic systems that act on a graph $\mathcal{G} = (V, E)$ and that synchronize. As a first example, we take a simple linear system, known as the Laplacian associated with the adjacency matrix of $\mathcal{G}$ the ODE on $\mathbb{R}^{|V|}$

$$\frac{d}{dt}x = L_{\mathcal{G}}(x), \tag{1}$$

where $|V| = n$ and $L_{\mathcal{G}}$ is the Laplacian matrix of the adjacency matrix of $\mathcal{G}$. Which can also be written as:

$$\frac{d}{dt}x_k = \sum_{j \in \mathcal{V}(k)} (x_j - x_k), \tag{2}$$

where x_k are the coordinates of the vector x and $\mathcal{V}(k)$ denotes the set of closest neighbors of vertex k.

This system is such that the diagonal

$$\Delta = \{x \in \mathbb{R}^n : x_i = x_j \ \forall\ 0 \le i \le j \le n\} \tag{3}$$

is a **global attractor**, that is, such that $x(t) \to \Delta$ as $t \to \infty$ for all initial condition $x \in \mathbb{R}^n$.

The second system that we analyze is the nonlinear system, known as the Kuramoto Model which is the ODE on $\mathbb{R}^{|V|}$

$$\frac{d}{dt}\varphi_k = \omega_k + \sigma \sum_{j \in \mathcal{V}(k)} \sin(\varphi_j - \varphi_k), \tag{4}$$

where $\mathcal{V}(k)$ denotes the set of closest neighbors of node k, the natural frequencies are distributed according to some probability density $\omega \mapsto g(\omega)$ and σ is the coupling strength with a suitable scale, so that the model has a good behavior when $|V| = n \to \infty$. The conditions under which we observe synchronization behaviors are well known.

Paths construction

For a fix a threshold $\epsilon > 0$, the ϵ-synchronized subnetwork G_x corresponding to configuration $x \in \mathbb{R}^n$ is defined by the adjacency matrix

$$A_x(i,j) = A(i,j)\chi(x)_{\{|x_i - x_j| \le \epsilon\}}. \tag{5}$$

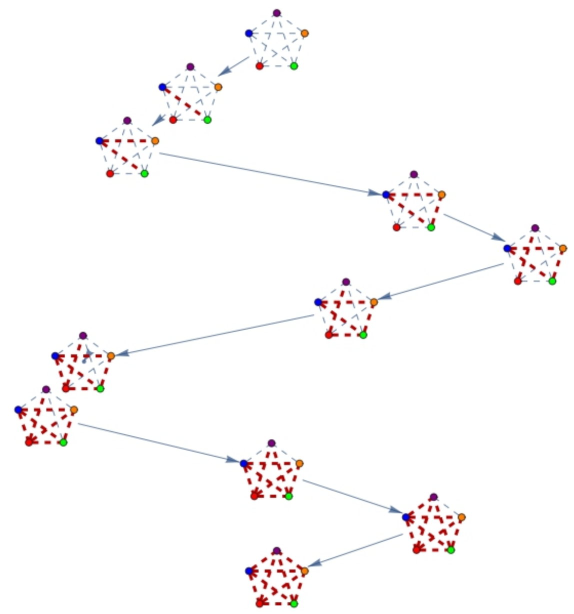
Otherwise said, the ϵ-synchronized subnetwork G_x is composed of the edges (i,j) of G for which $|x_i - x_j| \le \epsilon$. By hypothesis $G_{x(t)} \to G$ as $t \to \infty$, and since there is a finite number of subnetworks, and we are assuming global synchronization, then for each initial condition $x \in \mathbb{R}^n$ there exists a finite sequence of switching times $t_0 = 0 < t_1 < t_2 < \cdots < t_{N(x)}$ and corresponding sequence of ϵ-synchronized subnetworks $\mathcal{G}_x := (G_{x(0)}, G_{x(t_1)}, \ldots, G_{x(t_{N(x)})})$ such that

$G_{x(t_i)} \ne G_{x(t_{i+1})}$, for each $0 \le i < N(x)$, and

$G_{x(t)} = G_{x(t_i)}$ with $i = \max\{0 \le j \le N(x) : t \ge t_j\}$.

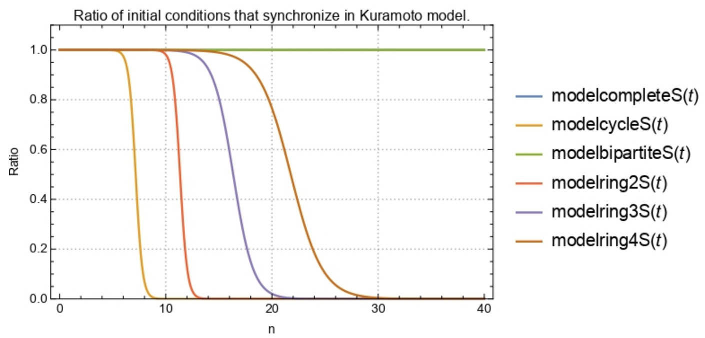
Example

We consider the Laplacian on K_5 for an initial condition x_0.



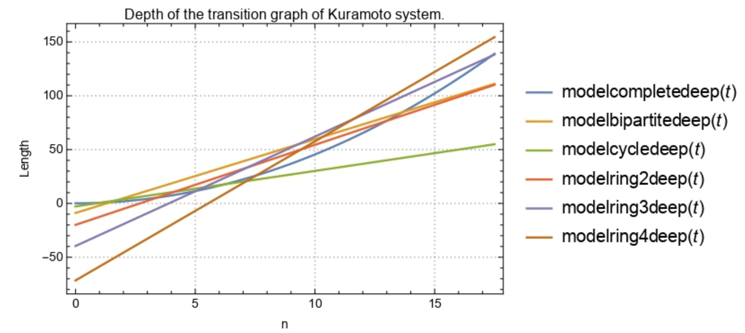
Analysis types:

- Ratio of initial conditions that synchronize.
  In the following figure we show a comparison of the proportion of initial conditions that synchronize in the Kuramoto system in each type of graph that we analyze, which depends on the dimension.



The following results that we will show are for initial conditions that, applying the Kuramoto system, synchronize.

- Depth.
  For each of the graph types that were analyzed, we looked for the depth of its transition graph, that is, the length of the longest path. In the next figure, we show the behavior of the growth of depth as the dimension of each graph n grows.



Below in the table, we show the comparison between linear and non-linear case, the pink boxes refer to the exactly same behaviors. In cases where an asterisk (*) appears, it is because it is faster. For the case of small world networks, as we said before, we did not find initial conditions that synchronize to do the analysis, that is why the empty sign is shown in the column of the non-linear case.

Graph		Property	Linear case	Non-linear case
K_n		Unfeasible subgraphs	Exp	Exp
		Depth	Polynomial	Polynomial
		Path length distribution	Normal	Normal
		Number of paths	Exp	Exp
$K_{n,n}$		Unfeasible subgraphs	Exp	Exp
		Depth	Linear	Linear*
		Path length distribution	Normal	Normal*
		Number of paths	Exp	Exp*
C_n		Unfeasible subgraphs	0	0
		Depth	Linear	Linear*
		Path length distribution	Normal	Normal*
		Number of paths	Exp	Exp*
$C(n,k)$	$p_r = 0$	Unfeasible subgraphs	Exp	Exp
		Depth	Linear	Linear*
		Path length distribution	Normal	Normal*
		Number of paths	Exp	Exp*
	$p_r = 1/3$	Unfeasible subgraphs	Exp	Exp
		Depth	Linear	$\emptyset$
		Path length distribution	Normal	$\emptyset$
		Number of paths	Exp	$\emptyset$
	$p_r = 2/3$	Unfeasible subgraphs	Exp	Exp
		Depth	Linear	$\emptyset$
		Path length distribution	Normal	$\emptyset$
		Number of paths	Exp	$\emptyset$
	$p_r = 1$	Unfeasible subgraphs	Exp	Exp
		Depth	Linear	$\emptyset$
		Path length distribution	Normal	$\emptyset$
		Number of paths	Exp	$\emptyset$

References

[1] A. Jamakovic & P. Van Mieghem, The Laplacian Spectrum of Complex Networks, *Proceedings of the European Conference on Complex Systems*, Oxford, September 25-29, (2006).

[2] F. A. Rodrigues, T. K. D. M. Peron, P. Ji, & J. Kurths, The Kuramoto model in complex networks, *Phys. Rep.*, **610**, 1 (2016).