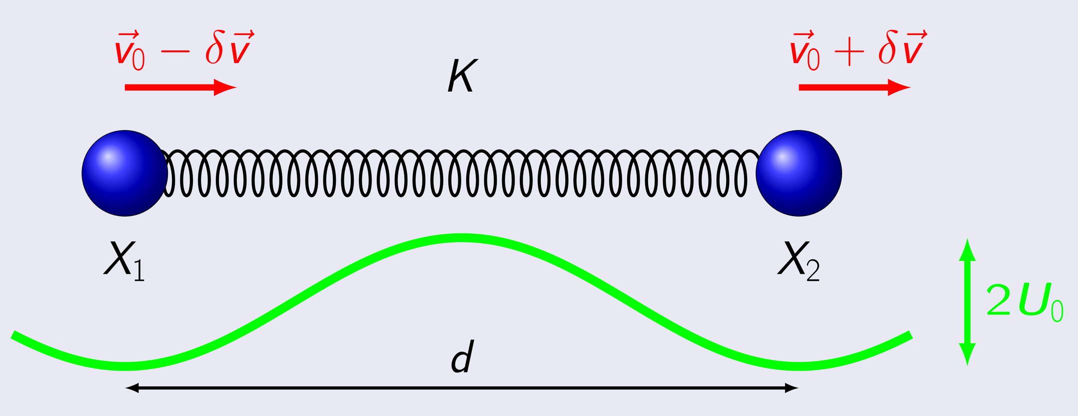
Instabilité paramétrique d'un système conservatif

Johann Maddi Michel Saint-Jean Christophe Coste

Université de Paris, Laboratoire Matière et Systèmes Complexes

System and equations of motion



Modal description :

$$\begin{cases} Y = (X_2 + X_1 - d)/2 \leftrightarrow \text{Translation} \\ X = (X_2 - X_1 - d)/2 \leftrightarrow \text{Vibration} \end{cases}$$

Equations of motion :

$$\begin{cases} \ddot{Y} = -U_0 \sin(Y)\cos(X) \\ \ddot{X} = -2KX - U_0\cos(Y)\sin(X) \end{cases}$$

Nonlinear coupling between the two modes induced by the underlying potential

Parametric instability - A naive approach by Mathieu equation

- Without translation of the center of mass :

$$Y = 0$$
$$\ddot{X} \simeq -\omega^2 X$$

- With translation of the center of mass :

$$Y \simeq C(v_0)\sin(\omega(v_0)t)$$
$$\ddot{X} \simeq -\omega^2(t)X = -\omega_0^2[1 + h(v_0)\cos(\omega(v_0)t)]X$$

Parametric excitation of the vibration mode

Mathieu equation :
- Highlights a possible parametric instability of X
- Does not include energy conservation

Multi-scale approach

Approximate solutions :

$$\begin{cases} Y \simeq \varepsilon Y_1 + \varepsilon^3 Y_3 + ... \\ X \simeq \varepsilon X_1 + \varepsilon^3 X_3 + ... \end{cases} \quad \text{with} \quad \begin{aligned} \varepsilon^2 &= v_0^2/4U_0 = E/4U_0 \\ Y(t) &= Y(t_0 \equiv t, t_2 \equiv \varepsilon^2 t) \end{aligned} \quad \text{same for } X(t)$$

Developed equations of motion :

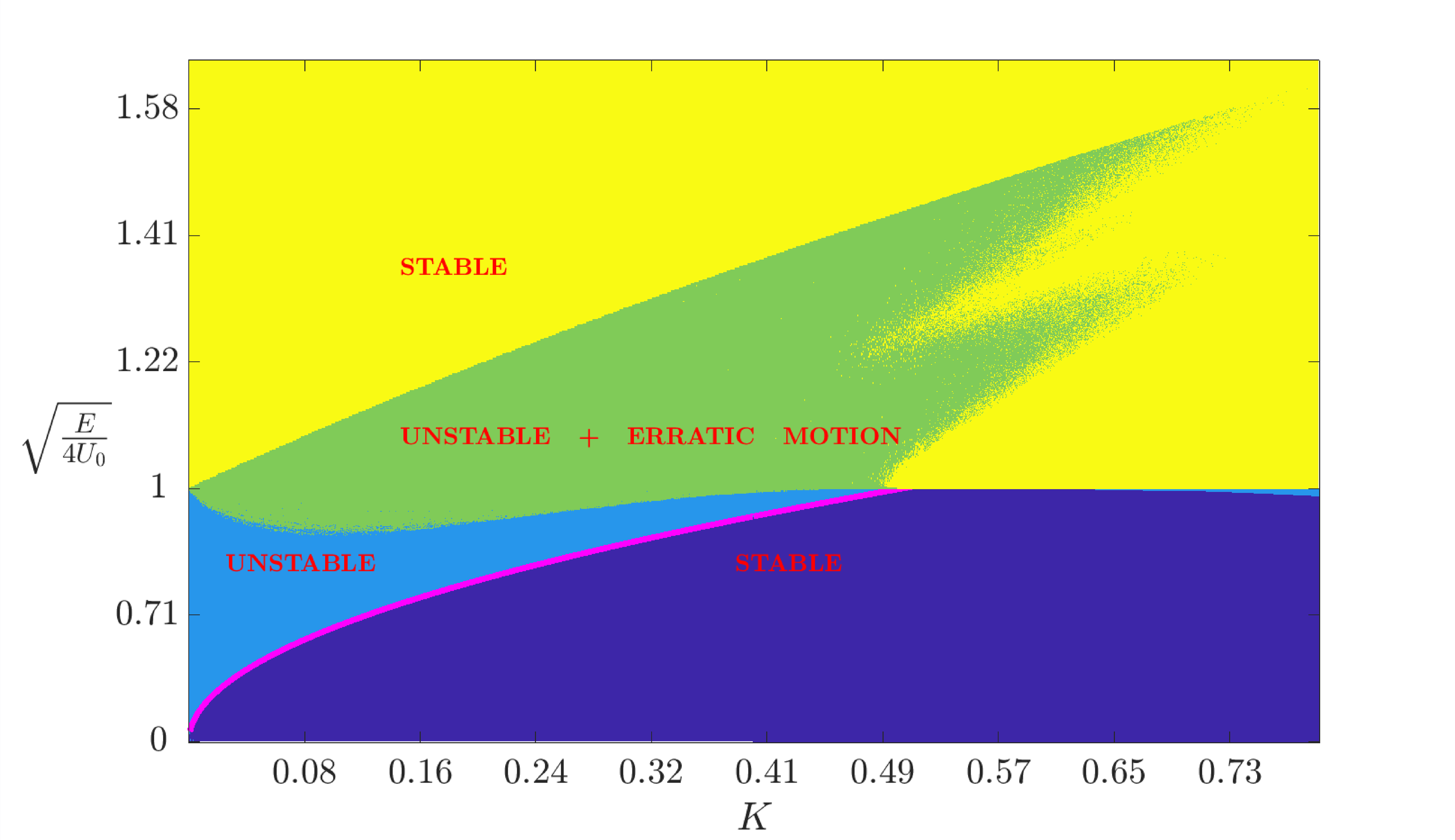
$$\begin{cases} \ddot{Y} = -Y + \frac{X^2Y}{2} + \frac{Y^3}{6} \\ \ddot{X} = -(2\tilde{K}+1)X + \frac{Y^2X}{2} + \frac{X^3}{6} \end{cases} \quad \text{with assumption} \quad K = \varepsilon^2\tilde{K}$$

This developement leads to equations of amplitudes and exhibits two constants of motion.

Constants of motion :

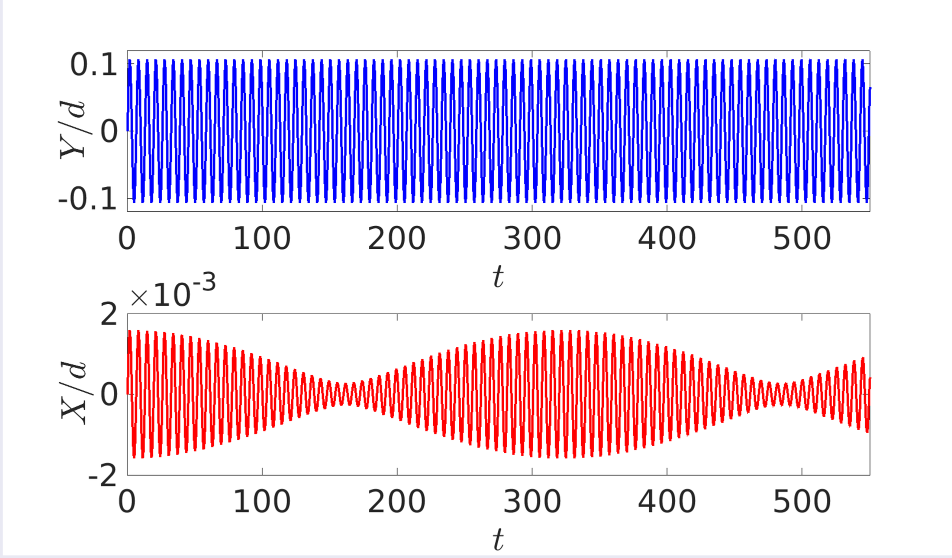
$$\begin{aligned} Y &\equiv a\cos(\phi) \\ X &\equiv b\cos(\psi) \\ \theta &= 2(\psi - \phi) \end{aligned} \quad \text{with} \quad \begin{aligned} a^2 + b^2 \equiv E &\longleftrightarrow \text{energy conservation} \\ \frac{a^4+b^4}{8} - \frac{a^2b^2}{4}\cos(\theta) - \tilde{K} = J &\longleftrightarrow \text{energy transfer} \end{aligned}$$

Phase diagram of the dimer

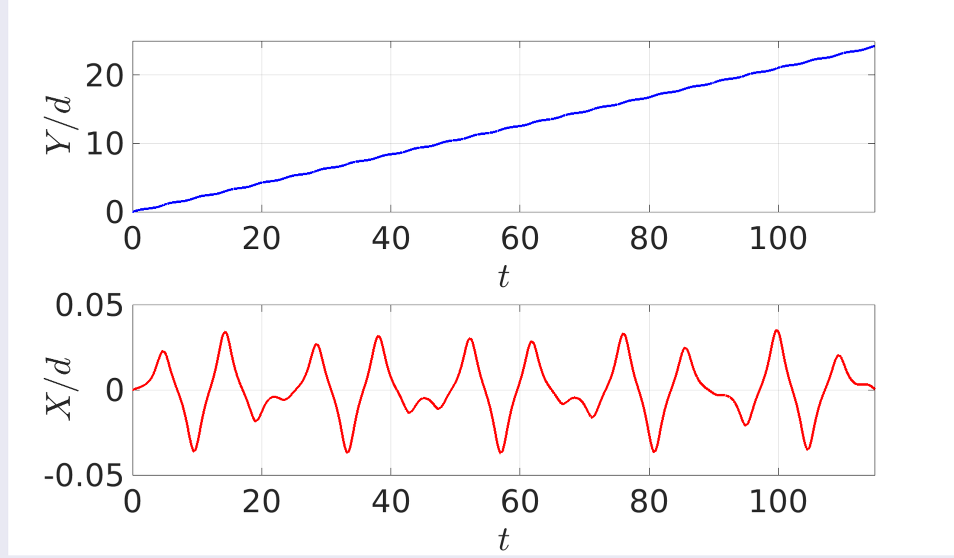


Stables regions

- $E < 4U_0$: The two modes behave like two weakly coupled oscillators.

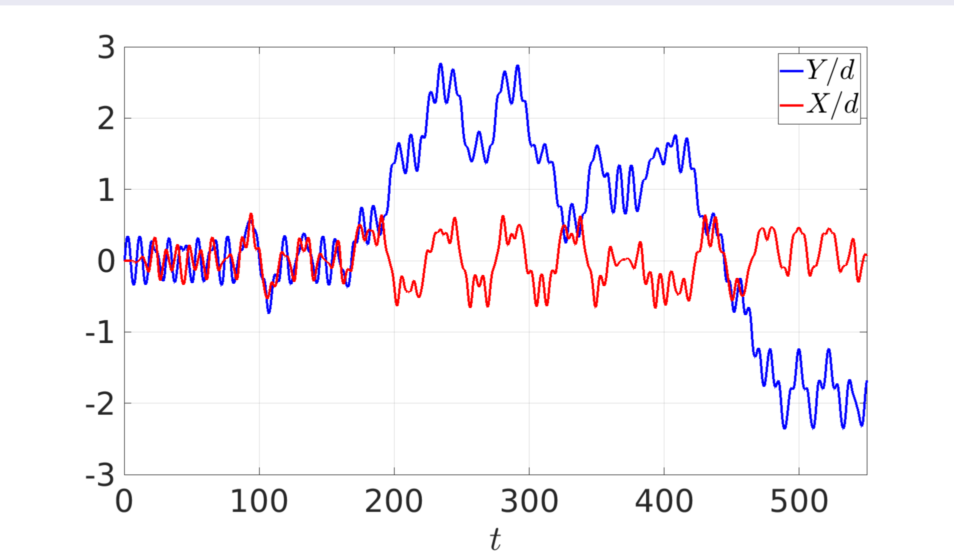


- $E > 4U_0$: The center of mass (Y) has sufficient energy to overcome the potential barrier and we observe a classic drift motion.

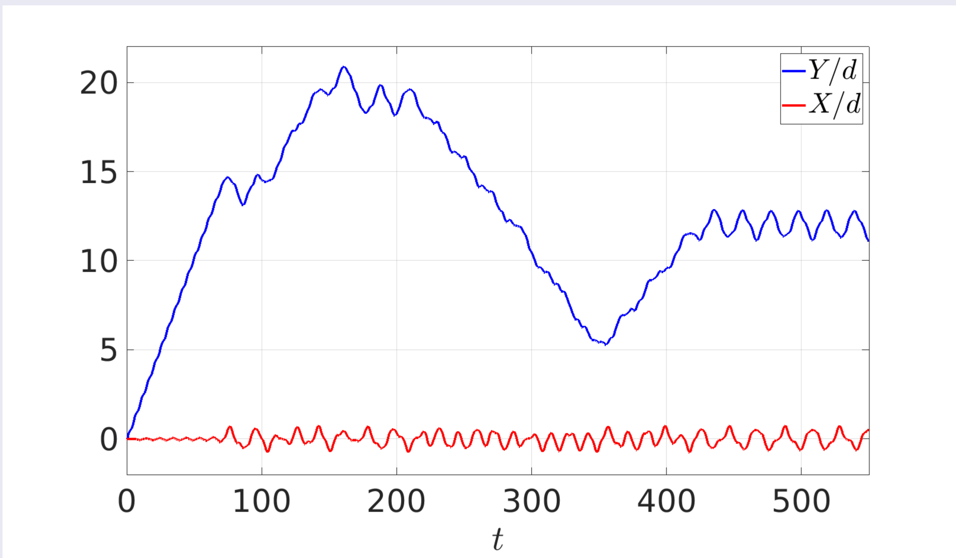


Unstable + Erratic motion

- $E < 4U_0$: Translation mode (Y) transfers sufficient energy to stretch the dimer over more than a half-period of the underlying potential. This stretching results in a possibility for the center of mass to overcome its potential barrier and then an erratic motion

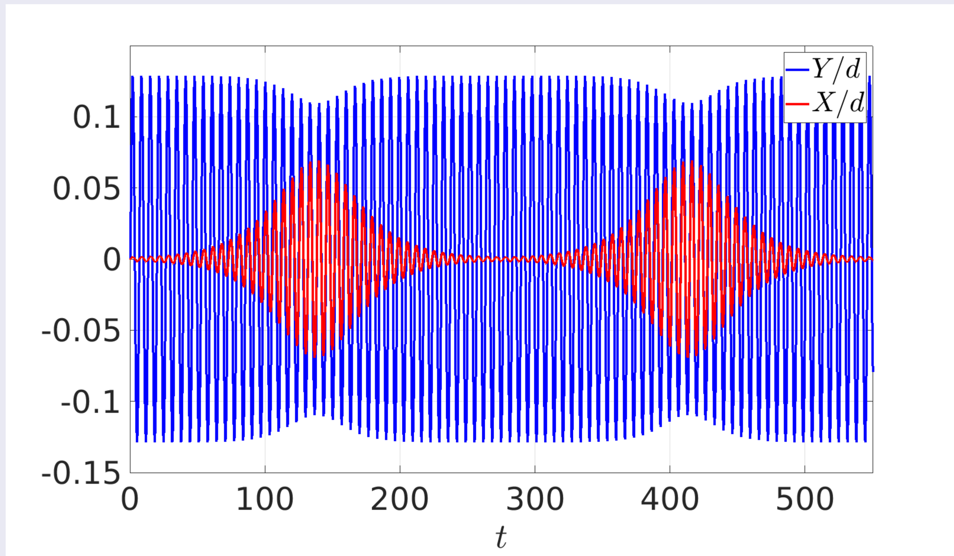


- $E > 4U_0$: The center of mass (Y) has sufficient energy to overcome the potential barrier. The addition of this drift motion and the parametric instability of the vibration mode results in an erratic motion



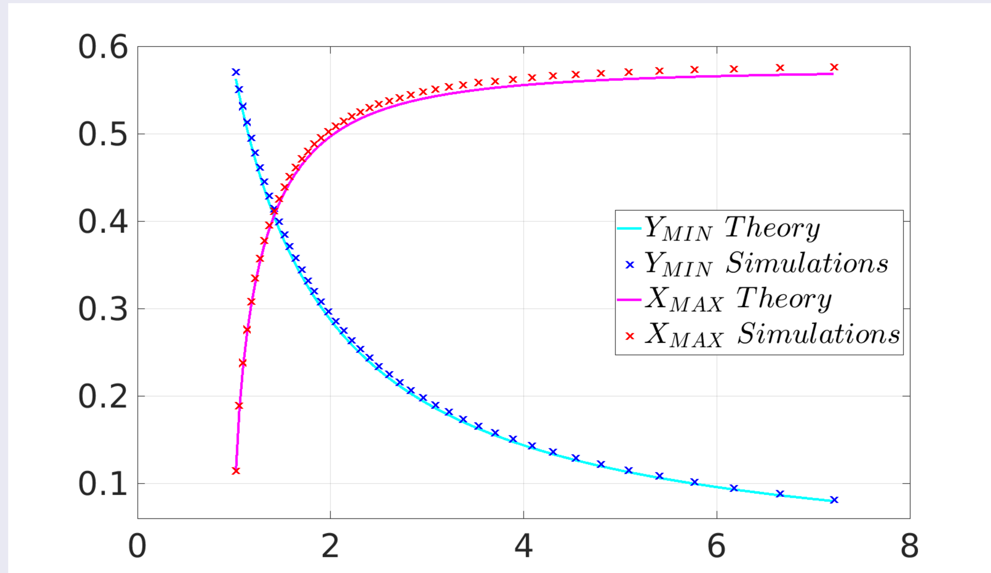
Instable region $E < 4U_0$:

Parametric excitation of translation mode (Y) on vibration mode (X) is sufficient to trigger parametric instability. This instability results in a saturated and periodic resonance of the dimer vibration.



Theoretical predictions

- **Boundary instability :** Numerical simulations and theoretical predictions show that the onset of instability occurs for $E > 8K$
- **Amplitude extrema :** The resonance stops when the instability condition is no longer fulfilled. The extrema Y_{min} and X_{max} of Y and X envelopes are given by :



$$Y_{min} = \sqrt{8K}$$
$$X_{max} = \sqrt{E - 8K}$$