Investigation of a Wave Diffusion Problem

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Several recent studies have shown that dispersive wave system, in presence of a weak (and stationary) random mean flow (of much smaller magnitude than the group velocity), may show a scattering behaviour (see [1] and [4]). Consistently with theoretical developments [2], this scattering only involves the wave number orientation and not its magnitude. Recently, it has been proposed that internal waves lead to the same result, with a scattering of wave number on the dispersion relation cone while the frequency remains constant [3] ; however, due to the particular nature of the dispersion relation, this result does not reflect the capability of the group velocity to evolve without changing the frequency, potentially breaking the assumption of a weak mean flow compared to the group velocity. Starting from a set of PDEs describing dispersive waves in presence of a mean flow, we first present a procedure using the Wigner transform to reduce the system to the following set, constituted of a Liouville equation on the energy density A with a Hamiltonian Ω involving a Doppler shifting term

$$\partial_t A + \{\Omega, A\} = 0, \tag{1}$$
$$\Omega = \omega + \mathbf{U} \cdot \mathbf{k}, \tag{2}$$

where ω is the intrinsic frequency, given by the dispersion relation

$$\omega = |k|^\alpha, \quad \alpha \in \mathbb{R}. \tag{3}$$

We identify two relevant parameters for our study : a first parameter, α, controlling the power law of the dispersion relation ; and a second parameter, ϵ, indicating the initial ratio between the average value of the stationary mean flow and the initial group velocity. We then investigate this phase diagram thanks to a 2D ray tracing scheme in order to identify the different asymptotic regimes. This scheme is derived from the Hamiltonian system

$$\partial_t \mathbf{x} = \mathbf{U}(\mathbf{x}) + \mathbf{c_g}(\mathbf{x}, \mathbf{k}), \tag{4}$$
$$\partial_t \mathbf{k} = -\mathbf{k} \cdot \nabla_{\mathbf{x}} \mathbf{U}(\mathbf{x}). \tag{5}$$

We demonstrate that, as shown in prior studies, there is a range of dispersive systems for which the frequency stays constant and diffusion is only observed in wave number angle. Moreover, we show that, for lower values of the power law controlling the dispersion relation, the assumption that the group velocity is larger than the mean flow can break, yielding to a diffusion to larger values in frequency, as well as in wave number.

Références

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