Inertia-gravity waves, a canonical example of nonlinear eigenvalue problems

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Global rotation and density stratification are ubiquitous in natural systems (e.g. in the Earth's oceans and atmosphere, in fluid planetary interiors or in stars). Consequently, geophysical flows are often shaped by the action of the so-called inertia-gravity waves (IGWs). From a mathematical viewpoint, IGWs offer a canonical example to investigate the physics of quadratic eigenvalue problems (QEP) of the form [1]

$$\lambda^2 \boldsymbol{u} + \lambda \mathcal{A}(\boldsymbol{u}) + \mathcal{B}(\boldsymbol{u}) = \boldsymbol{0},$$

where $[\mathcal{A}, \mathcal{B}]$ are two linear operators with specific symmetries, and with the eigenvalue $\lambda \in \mathbb{C}$. Here, we study the properties of IGWs that can exist in pancake-like stratified vortices, which are often generated by turbulence in geophysical environments. Typical applications are oceanic eddies (e.g. the Meddies) or Jovian vortices (e.g. the Great Red Spot in Jupiter). We consider a fluid enclosed within a rigid triaxial ellipsoid, which is stratified in density with a constant Brunt-Väisälä frequency (using the Boussinesq approximation) and uniformly rotating along a (possibly) tilted axis with respect to gravity. Despite the corresponding QEP is generically is an ill-posed Cauchy problem, we find that the spectrum is pure point in ellipsoids (i.e. only consists of eigenvalues) with smooth polynomial eigenvectors. Then, we fully characterise the spectrum using numerical calculations and microlocal analysis [3,4]. In addition to the usual IGWs (which exist in unbounded fluids), we report the existence of gravito-inertial (surface) waves whose energy is maximum near the boundary. These waves appear similar to the Kelvin waves in oceanography [5], and we show that they owe their existence to a non-trivial mathematical property satisfied by the pressure on the boundary [6]. Finally, we aim to explore whether parametric instabilities (e.g. the elliptical instability or the triadic resonant instability), which are also governed by a QEP in the linear theory, could sustain a transition towards (wave) turbulence in such stratified vortices.

References

1. E. M. Barston, *J. Math. Phys.*, **8** (3), 523-532, 1967
2. S. Vantieghem, *Proc R. Soc. A*, **470** (2168), 20140093, 2014
3. J. Vidal & Y. Colin de Verdière, *Proc. R. Soc. A*, In press, 2024
4. Y. Colin de Verdière & J. Vidal, *J. Spectr. Theory*, ArXiv:2305.01369, 2024
5. W. Thomson, *Proc. R. Soc. Edinburgh*, **10**, 92-100, 1880
6. Y. Colin de Verdière & J. Vidal, *J. Contemp. Math*, Preprint, 2024